NILPOTENT GROUPS AND THEIR GENERALIZATIONS*

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Nilpotent finite groups may be defined by a great number of properties. Of these the following three may be mentioned, since they will play an important part in this investigation. (1) The group is swept out by its ascending central chain (equals its hypercentral). (2) The group is a direct product of p-groups (that is, of its primary components). (3) If S and T are any two subgroups of the group such that T is a subgroup of S and such that there does not exist a subgroup between S and T which is different from both S and T, then T is a normal subgroup of S. These three conditions are equivalent for finite groups; but in general the situation is rather different, since there exists a countable (infinite) group with the following properties: all its elements not equal to 1 are of order a prime number p; it satisfies condition (3); its commutator subgroup is abelian; its central consists of the identity only.

A group may be termed soluble, if it may be swept out by an ascending (finite or transfinite) chain of normal subgroups such that the quotient groups of its consecutive terms are abelian groups of finite rank. A group satisfies condition (1) if, and only if, it is soluble and satisfies condition (3) (§2); and a group without elements of infinite order satisfies (1) if, and only if, it is the direct product of soluble p-groups (§3); and these results contain the equivalence of (1), (2) and (3) for finite groups as a trivial special case. If a group without elements of infinite order may be swept out by an ascending chain of subgroups such that each is a normal subgroup of the next one and such that the quotient groups of its consecutive terms are cyclic, then (2) and (3) are equivalent properties, though they no longer imply (1) (§4). If a group satisfies condition (1)—or suitable weaker conditions—then the elements of finite order in this group generate a subgroup without elements of infinite order which is a direct product of p-groups.

A seemingly only slightly stronger condition than (3) is the following property: (3′) If S and T are any two subgroups of the group such that T is a subgroup of S and such that there exists at most one subgroup between S and T which is different from both S and T, then T is a normal subgroup of S. Clearly (3′) implies (3), though there exist groups which satisfy (3), but not (3′). A closer investigation reveals however that (3′) is a much stronger imposition than it seems to be, since it is possible to prove the following theorem: A group, that either does not contain elements of infinite

* Presented to the Society, September 5, 1939; received by the editors August 28, 1939.

order or else is swept out by its ascending central chain, satisfies condition (3′) if, and only if, all its subgroups are normal subgroups, that is, if it is either abelian or hamiltonian (§6).

In an appendix we have given a list of properties which may serve as a definition of nilpotent groups, provided the group in question is finite; and since these properties are in general no longer equivalent, we have added a chart, indicating their interrelations.

0. In this section we state some notations and facts which will be used in the course of this investigation.

0.A. $Z(G)$ is the *central* of the group G.

$Z_0(G)=1$; $Z_\nu(G)\leqq Z_{\nu+1}(G)\leqq G$ and $Z(G/Z_\nu(G))=Z_{\nu+1}(G)/Z_\nu(G)$; if λ is a limit-ordinal, then $Z_\lambda(G)$ is the join of all the $Z_\nu(G)$ for $\nu<\lambda$, so that the ascending central chain $Z_\nu(G)$ is defined for every finite or infinite ordinal ν.

$C(G)$ is the *commutator subgroup* of G.

$C_0(G)=G$, $C_{i+1}(G)=C(C_i(G))$ so that $C_i(G)$ is defined for every integer i.

$C^0(G)=G$, $C^{i+1}(G)$ is the subgroup generated by all the elements $gcg^{-1}c^{-1}$ for g in G, c in $C^i(G)$, so that the descending chain $C^i(G)$ is defined for every integer i.

0.B. *Every subgroup of the group G is a normal subgroup of G* if, and only if, $x^{-1}yx$ is a power of y for every pair x, y of elements in G. If every subgroup of G is a normal subgroup of G, then there are two possibilities:

(A) G is abelian, that is, $C(G)=1$, $Z(G)=G$.

(H) G is hamiltonian. Then $C(G)$ is of order 2, $Z(G)$ consists of those elements whose order is not divisible by 4, $G/Z(G)$ is a direct product of two cyclic groups of order 2, so that $C_2(G)=C^2(G)=1$ and $G=Z_2(G)$. Furthermore G is a direct product of three groups Q, T, U where Q is a quaternion group, T an abelian group all of whose elements not equal to 1 are of order 2, and U an abelian group all of whose elements are of odd order so that in particular G does not contain elements of infinite order.

0.C. If p is a prime number, then a group G is said to be a *p-group* if all its elements are of order a power of p, and *primary components* are [greatest] p-subgroups which contain all the elements of order a power of p.

0.D. If A is an *abelian group*, then its elements of finite order form a subgroup $F(A)$ which is the direct product of its primary components; and $A/F(A)$ does not contain any element not equal to 1 of finite order. The group A may now be said to be of *finite rank*, if

(i) the elements in A which satisfy $x^p=1$, p a prime number, form a finite group,

(ii) there exists a finite number of elements $a(1), \cdots, a(n)$ in A so that

for every element x in A for which $x \not\equiv 1$ modulo $F(A)$ there exist integers $m \neq 0$, $m(i)$ satisfying $\prod_{i=1}^{n} a(i)^{m(i)} \equiv x^m$ modulo $F(A)$.

If $F(A) = 1$, that is, if the identity is the only element of finite order in the abelian group A, then the smallest integer n, meeting the requirement (ii), is termed the *rank* of A. The rank of such a group A is at the same time the greatest number of linearly independent elements in A. If B is a proper subgroup of A, then either the rank of B is smaller than the rank of A; or if A and B have the same rank, then there exists an element w in A, but not in B, and a positive integer i so that w^i is an element in B.

0.E. If G is a *group with abelian central quotient group*, then $C(G) \leqq Z(G)$; and all the properties we are going to use are simple consequences of the following well known formulas:

$$x(yz)x^{-1}(yz)^{-1} = (xyx^{-1}y^{-1})(xzx^{-1}z^{-1}),$$
$$(xy)^i = (yxy^{-1}x^{-1})^{i(i-1)2^{-1}}x^iy^i.$$

0.F. In the introduction we have mentioned a condition, concerning greatest subgroups, which will play a fundamental part in the course of our investigations. For future reference this property may be restated here as follows.

(G) *If S and T are any two subgroups of the group G so that $S < T$ and so that there does not exist any subgroup B between S and T which is different from both S and T, then S is a normal subgroup of T.*

We mention furthermore that the subgroup S of T is said to be a *greatest subgroup* of T, whenever S is a proper subgroup of T and there does not exist any subgroup between S and T which is different from both S and T. Thus (G) implies, in short, that greatest subgroups are normal subgroups.

1. In this section we are going to prove several auxiliary theorems which will be needed in the future. Most of them are concerned with the existence of central elements not equal to 1 which lie in prescribed subgroups, provided the group satisfies the condition (G), enunciated in 0.F.

The crosscut of all the greatest subgroups of a group G is always a characteristic and, therefore, a normal subgroup $M(G)$ of G.

(1.1) *If there exist greatest subgroups of the group G, and if every greatest subgroup of G is a normal subgroup of G, then $G/M(G)$ is an abelian group not equal to* 1.

For if S is a greatest subgroup of G, and if at the same time S is a normal subgroup of G, then G/S is a cyclic group of order a prime number, so that $C(G) \leqq S$.

(1.2) *If the group G is different from* 1, *if G is generated by a finite number of elements, and if every greatest subgroup of G is a normal subgroup of G, then* $G \neq C(G)$.

Proof. Since G is generated by a finite number of elements, there exists a smallest set N of generators of G (so that no proper subset of N generates G). If u is some element in N, then there exists a greatest subgroup U of G which does not contain u but contains all the other elements in N, since the subgroup of G which is generated by the elements different from u in N is different from G and hence cannot contain u. If V is a subgroup of G such that $U < V$, then V contains u and is therefore equal to G; that is, U is a greatest subgroup of G. Hence $M(G) < G$. But it follows from (1.1) that $C(G) \leqq M(G)$, and this shows that $C(G) < G$.

(1.3) *If G is generated by a finite number of elements, if C is a normal subgroup of G, and if G/C is finite, then C is generated by a finite number of elements.*†

REMARK. That the hypothesis of the finiteness of G/C is really needed may be seen from the fact that the commutator subgroup of a free group with two generators is not generated by a finite number of elements.

Proof. Denote by T some finite set of generators of G and by R a complete set of representatives of G/C so that both R and T are finite sets. If X is any element in G/C, then denote by $r(X)$ the uniquely determined element in R which satisfies $X = Cr(X)$. The elements

$$(X, Y) = r(XY)r(Y)^{-1}r(X)^{-1}$$

for X and Y in G/C and the elements

$$s(t) = tr(Ct)^{-1}$$

for t in T are in the normal subgroup C of G and so are the elements $r(X)s(t)r(X)^{-1}$. Since R and T are finite sets, it follows that the set D consisting of all the elements (X, Y) and $r(X)s(t)r(X)^{-1}$ for X, Y in G/C and t in T is a finite subset of C. Let D^* be the subgroup generated by D.

D^* is clearly a subgroup of C; and G is generated by adjoining the elements $r(X)$ to D^*, since $t = s(t)r(Ct)$. We have

$$\begin{aligned} r(X)(Y, Z)r(X)^{-1} &= r(X)r(YZ)r(Z)^{-1}r(Y)^{-1}r(X)^{-1} \\ &= (X, YZ)r(XYZ)r(Z)^{-1}r(XY)^{-1}(X, Y)^{-1} \\ &= (X, YZ)(XY, Z)^{-1}(X, Y)^{-1}, \end{aligned}$$

† The proof is an obvious application of the Schreier-Reidemeister method.

so that every transform of some (Y, Z) by some $r(X)$ is in D^*. Also

$$r(X)r(Y)s(t)r(Y)^{-1}r(X)^{-1} = (X, Y)r(XY)s(t)r(XY)^{-1}(X, Y)^{-1},$$

so that every transform of any element in D by any of the $r(X)$ is in D^*. Since G is generated by adjoining the elements $r(X)$ to D^*, this shows that D^* is a normal subgroup of G. Since the (X, Y) are in D^*, and since D^* is a subgroup of C, the finite groups G/C and G/D^* are essentially the same; and this implies $D^*=C$. Thus it has been proved that the finite set D of elements is a set of generators of C.

LEMMA 1.4.† *If G is generated by a finite number of elements, if G does not contain elements of infinite order, then every $C_i(G)$ is generated by a finite number of elements; and if, furthermore, condition* (G) *is satisfied by G, then $C_i(G) \neq 1$ implies $C_{i+1}(G) < C_i(G)$.*

Proof. $C_0(G)=G$ is generated by a finite number of elements. Hence assume that it has already been shown that $C_i(G)$ is generated by a finite number of elements. Then $C_i(G)/C_{i+1}(G)$ is generated by a finite number of elements and all its elements are of finite order, since all the elements in $C_i(G) \leqq G$ are of finite order. Hence $C_i(G)/C_{i+1}(G)$ is finite. Since $C_{i+1}(G) = C(C_i(G))$, it follows now from (1.3) that $C_{i+1}(G)$ is generated by a finite number of elements; and hence every $C_j(G)$ is generated by a finite number of elements. It is now a consequence of (1.2) that $C_i(G) \neq 1$ together with condition (G) imply $C_{i+1}(G) < C_i(G)$.

(1.5) *If every greatest subgroup of G is a normal subgroup of G, if the commutator subgroup of G is abelian, if G is generated by a pair of elements a, b so that b is in the commutator subgroup of G, then G is cyclic.*

Proof. G may be generated by adjoining a to $C(G)$. If $c=aba^{-1}b^{-1}$, then the elements a^ica^{-i} for integral i generate a normal subgroup of G which is contained in $C(G)$—since b permutes with every element in $C(G)$—and modulo which G is abelian, since it contains c, and since G is generated by a and b. Hence $C(G)$ is generated by the elements a^ica^{-i}. This implies however that G is generated by a and c. If c is not a power of a, then there exists a greatest subgroup of G which contains a, but not c. This subgroup V is a greatest subgroup of G so that $C(G) \leqq V$. Since V contains both a and $C(G)$, it follows that $V=G$ so that V would contain c in contradiction to the choice of V. Hence c is a power of a so that G is the cyclic group, generated by a.

(1.6) *Suppose that the commutator subgroup of the group G is abelian and that G is generated by two elements a, b where $bc=cb$ for every c in $C(G)$. Let*

† Cf. Corollary 3.7 below which improves this result in certain respects.

$b_0=b$, $b_{i+1}=ab_ia^{-1}b_i^{-1}$; and let $G_0=G$ and G_{i+1} be generated by adjoining a to $C(G_i)$. Then G_i is generated by a and b_i, $C(G_i)$ by the elements $a^jb_{i+1}a^{-j}$ for integral j, and $C(G_i)=C^{i+1}(G)$.

Proof. Since $G_0=G$, $b_0=b$, G_0 is generated by a and b_0. Assume now that G_i is generated by a and b_i. Let V_i be the subgroup generated by the elements $a^jb_{i+1}a^{-j}$ for integral j. Then $V_i\leqq C(G_i)$. Since b permutes with every element in $C(G)$ and therefore with every element in V_i, since the set of generators of V_i is transformed into itself by a, and since G is generated by a and b, it follows that V_i is a normal subgroup of G and therefore of G_i. Since finally G_i/V_i is abelian, as b_{i+1} is in V_i, it follows that $C(G_i)\leqq V_i$, that is, $C(G_i)=V_i$. Since G_{i+1} is generated by adjoining a to $C(G_i)=V_i$, it follows that G_{i+1} is generated by a and b_{i+1}.

Since $C(G_0)=C(G)=C^1(G)$, we may assume that $V_i=C(G_i)=C^{i+1}(G)$. Since b permutes with every element in $C^{i+1}(G)$, it follows that $C^{i+2}(G)$ is generated by the elements $a^jxa^{-j}x^{-1}$ for integral j and x in $C^{i+1}(G)$. Since $C^{i+1}(G)$ is an abelian group which is generated by the elements $a^rb_{i+1}a^{-r}$ for integral r, it follows that $C^{i+2}(G)$ is generated by the elements $a^ra^sb_{i+1}a^{-s}b_{i+1}^{-1}a^{-r}$ for integral r and s. If $s<0$, then

$$\begin{aligned}(a^sb_{i+1}a^{-s}b_{i+1}^{-1})^{-1} &= b_{i+1}a^sb_{i+1}^{-1}a^{-s}\\ &= a^s(a^{-s}b_{i+1}a^sb_{i+1}^{-1})a^{-s};\end{aligned}$$

and if $0\leqq s$, then

$$\begin{aligned}a^{s+1}b_{i+1}a^{-1-s}b_{i+1}^{-1} &= a^sab_{i+1}a^{-1}b_{i+1}^{-1}a^{-s}a^sb_{i+1}a^{-s}b_{i+1}^{-1}\\ &= a^sb_{i+2}a^{-s}a^sb_{i+1}a^{-s}b_{i+1}^{-1}\end{aligned}$$

and it follows by complete induction with regard to s that we have

$$a^sb_{i+1}a^{-s}b_{i+1}^{-1}=\prod_{j=0}^{s-1}a^jb_{i+2}a^{-j} \qquad \text{for } 0<s.$$

Hence $C^{i+2}(G)\leqq V_{i+1}=C(G_{i+1})\leqq C^{i+2}(G)$, and this completes the proof.

LEMMA 1.7. *If the commutator subgroup $C(G)$ of the group G is abelian, if G is generated by two elements u and v so that $vc=cv$ for every c in $C(G)$, and if G satisfies* (G), *then either $C^i(G)=1$ or $C^{i+1}(G)<C^i(G)$.*

Proof. As in (1.6) we use the following notations: $v_0=v$, $v_{i+1}=uv_iu^{-1}v_i^{-1}$; $G_0=G$, G_{i+1} is the subgroup generated by adjoining u to $C(G_i)$. Then it follows from (1.6) that *G_i is generated by u and v_i, and that $C(G_i)=C^{i+1}(G)$.*

Assume now that for some integer i we have $C^i(G)=C^{i+1}(G)$. Then $C(G_{i-1})=C(G_i)$ and therefore $G_i=G_{i+1}$. Hence G_i is generated by u and v_{i+1},

where v_{i+1} is contained in the abelian commutator subgroup $C(G_i) \leqq C(G)$ of G_i. Since G satisfies (G), so does G_i. Hence it follows from (1.5) that G_i is a cyclic group. Consequently $C(G_i)=1$, $v_{i+1}=1$, so that G_i is the cyclic group generated by u. Thus it follows that $1=C(G_i)=C^{i+1}(G)=C^i(G)$, and this completes the proof.

(1.8) *If B is an abelian group all of whose elements not equal to* 1 *are of the same order, if B possesses a countable basis* $\cdots, b(i), b(i+1), \cdots$, *and if the group G is generated by adjoining an element g to B that is subject to the relation $g^{-1}b(i)g=b(i+1)$, then condition* (G) *is not satisfied by the group G.*

Proof. B is a normal subgroup of G, and G/B is an infinite cyclic group. The elements not equal to 1 in B are either all of order a prime number p or they are all of infinite order. Accordingly three cases are distinguished.

CASE 1. The elements not equal to 1 in B are of order 2.

Denote by U the subgroup of B which is generated by all the elements $b(i)b(i+1)b(i+2)$, and by S the subgroup of G which is generated by $b(1)b(2)b(3)$ and g. Then U is the crosscut of B and S.

If $x \neq 1$ is any element in U, then it begins with some $b(i)$ and ends with some $b(i+j)$ where $1<j$. Thus U does not contain $b(1)b(2)$. Since $b(1)gb(1)^{-1}=b(1)b(2)g$, this shows that S is not a normal subgroup of G.

Suppose now that $S<T \leqq G$. Then T contains elements w which are in B but not in U. Such an element has a certain "length," so that T contains an element w in B but not in U which is of shortest length. Suppose that w begins with $b(i)$ and ends with $b(i+j)$. Since U contains $b(i+j-2)b(i+j-1)b(i+j)$, it follows that $0 \leqq j \leqq 1$. For $j=1$ we would find however that $b(i)b(i+1)b(i)b(i+1)b(i+2)=b(i+2)$ would be of shorter length. Hence $w=b(i)$ and $T=G$, so that S is a greatest subgroup of G though not a normal one.

CASE 2. The elements not equal to 1 in B are of order a prime number $p \neq 2$.

Denote by U the subgroup of B which is generated by the elements $b(i)b(i+1)$ and by S the subgroup of G which is generated by g and $b(1)b(2)$. Then U is the crosscut of S and B.

Any element $x \neq 1$ in U contains at least two elements $b(i)$ and $b(j) \neq b(i)$ as factors. Thus $b(1)$ is no element in U. Since 2 is relatively prime to p, it follows that $b(1)^2=b(1)b(2)^{-1}b(1)b(2)$ is not an element in U. Hence $b(1)b(2)^{-1}$ is not contained in U. Since $b(1)gb(1)^{-1}=b(1)b(2)^{-1}g$, this shows that S is not a normal subgroup of G.

Suppose now that $S<T \leqq G$. Then T contains elements w which are contained in B but not in U. Amongst these there are some of shortest "length"; and since U contains $b(i)b(i+1)$, it follows that such an element of shortest

length has the form $b(i)^n$ where n is relatively prime to p. Hence T contains $b(i)$ and therefore every $b(j)$, so that $T=G$. Thus S is a greatest subgroup of G but not a normal one.

CASE 3. The elements not equal to 1 in B are of infinite order.

Then let p be some prime number. Clearly the subgroup B^p which is generated by the elements $b(i)^p$ is a normal subgroup of G. The group G/B^p is then just of the type discussed under Cases 1 and 2. Hence G/B^p does not satisfy (G), so that G itself cannot satisfy (G).

LEMMA 1.9. *If condition* (G) *is satisfied by the group G, if $H\neq 1$ is a normal subgroup of G, if H is abelian and G/H cyclic, then the crosscut of $Z(G)$ and H is different from* 1.

Proof. Let g be an element that generates G modulo H. Then the crosscut of $Z(G)$ and H consists of exactly those elements x in H which satisfy $xg=gx$.

CASE 1. H contains elements not equal to 1 of finite order.

Then H contains an element u of order a prime number p. Denote by U the subgroup generated by u and g. The crosscut V of U and H is generated by the elements $u(i)=g^{-i}ug^i$ for integral i.

If there did not exist any relation between the elements $u(i)$, then U would be a group of the type discussed in (1.8). This is impossible, since G and U satisfy condition (G). Hence there exists a relation between the $u(i)$. This proves that V is generated by a finite number of elements, so that V is a finite abelian group not equal to 1.

There is nothing to prove if $V\leqq Z(U)$. If this is not the case, then $1<C(U)\leqq V$. Since V is finite, it follows that the chain of subgroups $C^i(U)$ ends, so that there exists an integer i satisfying $C^{i+1}(U)=C^i(U)<C^{i-1}(U)\leqq V$. Since G and U satisfy (G), it follows from Lemma 1.7 that $C^i(U)=1$. Hence $1<C^{i-1}(U)\leqq Z(U)$ so that the crosscut of $Z(U)$ and V is different from 1. Thus the crosscut of $Z(G)$ and H is different from 1.

CASE 2. All the elements not equal to 1 in H are of infinite order.

Then there exists for every element x in H and for every integer n at most one element y in H so that $x=y^n$.

Let u be an element not equal to 1 in H. Denote by U the subgroup generated by u and g. The crosscut V of U and H is generated by the elements $b(i)=g^{-i}ug^i$ for integral i.

If there did not exist any relation between the elements $b(i)$, then U would be of the type discussed in (1.8). This is impossible, since G and U satisfy condition (G). Hence there exists some relation between the $b(i)$, and we may assume without loss in generality that this relation has the following form:

$$(\alpha) \qquad \prod_{i=1}^{n-1} b(i)^{m(i)} = b(n)^m \qquad \text{with } m(1)m \neq 0;$$

though there does not exist any relation between $b(1), \cdots, b(n-1)$. Denote now by p some prime number that does not divide m or $m(1)$ and denote by V^p the subgroup of V which is generated by the elements x^p for x in V. Clearly V^p is a normal subgroup of U.

One verifies that $b(n+i)^{m^{i+1}}$ for $0<i$ is contained in the subgroup generated by $b(1), \cdots, b(n-1)$. Since

$$b(0)^{m(1)} = \prod_{i=2}^{n-1} b(i-1)^{-m(i)} b(n-1)^m,$$

as follows on transforming (α) by g^{-1}, one verifies too that $b(0)^{m(1)}$, $b(-i)^{m(1)^{i+1}}$ (for $0<i$) are elements in the subgroup generated by $b(1), \cdots, b(n-1)$.

Suppose now that $r=\prod_{i=1}^{n-1} b(i)^{r(i)} \equiv 1$ modulo V^p. Hence there exists an element s in V so that $r=s^p$. Then s may be represented as a product of elements $b(j)$ with $-i \leqq j \leqq i$ for $0<i$. Thus $s^{m(1)^{i+1}m^{i+1}}$ is certainly an element in the subgroup generated by $b(1), \cdots, b(n-1)$. If we put $t=m(1)^{i+1}m^{i+1}$, then t is relatively prime to p; and $r^t=s^{tp}$. Suppose that $s^t=\prod_{i=1}^{n-1} b(i)^{s(i)}$. Then

$$\prod_{i=1}^{n-1} b(i)^{tr(i)} = \prod_{i=1}^{n-1} b(i)^{ps(i)}.$$

Since there does not exist any relation between $b(1), \cdots, b(n-1)$, it follows that $tr(i)=ps(i)$. Since t is relatively prime to p, this implies that $r(i) \equiv 0$ modulo p. Thus we have proved that $b(1), \cdots, b(n-1)$ *form a basis of* V *modulo* V^p. This shows in particular that $V/V^p \neq 1$ is a finite abelian group. Hence it follows from what has been proved under Case 1 that the crosscut of $Z(U/V^p)$ and V/V^p is different from 1, since G and therefore U/V^p satisfy (G), since U/V is cyclic, and since V/V^p is abelian.

The automorphism which is induced by g in V/V^p has therefore fixed-elements not equal to 1; and it may be represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdots & 0 & 1 \\ m(1)m^{-1} & m(2)m^{-1} & m(3)m^{-1} & \cdot & \cdots & m(n-2)m^{-1} & m(n-1)m^{-1} \end{pmatrix}$$

so that the determinant

$$\begin{vmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ m(1)m^{-1} & m(2)m^{-1} & m(3)m^{-1} & \cdot & \cdots & m(n-2)m^{-1} & m(n-1)m^{-1}-1 \end{vmatrix} = D,$$

whose coefficients are rational numbers with denominators prime to p, is divisible by p; that is, the integer Dm^{n-1} is divisible by p. Since this holds true for every prime number p that does not divide m or $m(1)$, and since there exists an infinity of such prime numbers, it follows that $D=0$. Consequently there exists an element different from 1 in V which is a fixed-element under the automorphism induced by g in V, so that the crosscut of $Z(U)$ and V is different from 1. It follows that the crosscut of $Z(G)$ and H is different from 1.

LEMMA 1.10. *If the normal subgroup H of the group G is different from* 1, *if H is abelian and of finite rank, if there exists for every subgroup $K\neq 1$ of H and for every element g in G such that $gK=Kg$ an element $h\neq 1$ in K such that $gh=hg$, and if there exists an ascending chain* of subgroups B_ν such that*

(1) $H=B_2$;

(2) *B_ν is a normal subgroup of $B_{\nu+1}$;*

(3) *$B_{\nu+1}/B_\nu$ is abelian;*

(4) *B_ν is for limit-ordinals ν the set of all the elements contained in subgroups B_μ for $\mu<\nu$;*

(5) $G=B_\tau$,

then the crosscut of H and $Z(G)$ is different from 1.

Proof. It is possible to well-order the elements in G in such a way: $g(1), g(2), \cdots, g(\kappa), g(\kappa+1), \cdots$ that all the elements in B_ν precede all the elements that are not contained in B_ν.

If H contains elements of finite order that are different from 1, then there exists a prime number p such that H contains elements of the exact order p. The subgroup generated by these elements is finite, since H is of finite rank; and this subgroup shall be denoted by B_1. If however H does not contain elements of finite order which are different from 1, then we put $H=B_1$ so that B_1 is an abelian group different from 1 which is either finite or does not contain any element of finite order except 1 and which is a normal subgroup of G, since it is a characteristic subgroup of the normal subgroup H of G.

Denote by $H(\nu)$ the set of elements x in B_1 which satisfy $g(\nu)x=xg(\nu)$.

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* If we use the terminology introduced in §3, this amounts to saying that G/H is metacyclic.

Then it follows from our hypothesis concerning H and G that every $H(\nu)$ is a subgroup, different from 1, of B_1.

If $1<\mu$, then denote by $K(\mu)$ the crosscut of all the $H(\nu)$ for $\nu<\mu$. The $K(\mu)$ are subgroups of B_1 too.

If B_1 is finite, then the $K(\mu)$ form a descending chain of finite abelian groups. If B_1 is infinite, then 1 is the only element of finite order in B_1 so that there exists to every integer i and to every b in B_1 at most one solution x of $x^i=b$ in B_1. If g is an element in G, b an element in B_1, i an integer, then it follows that $gb^i=b^ig$ implies $b^i=gb^ig^{-1}=(gbg^{-1})^i$ and hence $gb=bg$ too. The $K(\mu)$ form therefore a descending chain of subgroups of the group B_1 of finite rank with the property that $K(\rho)<K(\sigma)$ if, and only if, the rank of $K(\rho)$ is smaller than the rank of $K(\sigma)$.

Thus it follows that there exists in both cases a smallest ordinal τ so that $K(\tau)=K(\nu)$ for $\tau<\nu$, that is, $K(\tau)\leqq Z(G)$. Since $K(\tau)<K(\mu)$ for $\mu<\tau$, it follows that τ cannot be a limit-ordinal so that $\tau=\rho+1$, and so that $K(\rho)\neq 1$. There exists furthermore a uniquely determined ordinal δ so that $g(\rho)$ is an element in $B_{\delta+1}$, but not in B_δ. Then all the $g(\nu)$ for $\nu<\rho$ are contained in $B_{\delta+1}$, so that all the commutators $g(\nu)^{-1}g(\rho)^{-1}g(\nu)g(\rho)$ are elements in B_δ; that is, they are elements $g(\mu)$ for $\mu<\rho$. If x is an element in $K(\rho)$, and if $\nu<\rho$, then

$$\begin{aligned} g(\nu)g(\rho)xg(\rho)^{-1} &= g(\rho)g(\nu)g(\nu)^{-1}g(\rho)^{-1}g(\nu)g(\rho)xg(\rho)^{-1} \\ &= g(\rho)g(\nu)xg(\nu)^{-1}g(\rho)^{-1}g(\nu)g(\rho)g(\rho)^{-1} \\ &= g(\rho)xg(\nu)g(\nu)^{-1}g(\rho)^{-1}g(\nu)g(\rho)g(\rho)^{-1} \\ &= g(\rho)xg(\rho)^{-1}g(\nu) \end{aligned}$$

so that $g(\rho)K(\rho)=K(\rho)g(\rho)$. Hence it follows from $K(\rho)\neq 1$ that there exists an element $v\neq 1$ in $K(\rho)$ satisfying $g(\rho)v=vg(\rho)$ and this proves that $K(\tau)\neq 1$. Since $K(\tau)\leqq Z(G)$ and since $K(\tau)\leqq B_1\leqq H$, it follows that the crosscut of H and $Z(G)$ is different from 1.

LEMMA 1.11. *Suppose that the property* (P) *of groups satisfies the following conditions*:

(i) *If* (P) *is satisfied by the group* H, *then* (P) *is satisfied by every quotient group of* H;

(ii) *If* (P) *is satisfied by the group* $H\neq 1$, *then* $Z(H)\neq 1$.

Then there exists for every group G, *satisfying* (P), *an ordinal* ζ *such that* $G=Z_\zeta(G)$.

Proof. If (P) is satisfied by the group G, then (P) is satisfied by $G/Z_\nu(G)$. If $Z_\nu(G)<G$, then it follows from (ii) that the central of $G/Z_\nu(G)$ is different

from 1, that is, $Z_\nu(G) < Z_{\nu+1}(G)$. Since the $Z_\nu(G)$ form an ascending chain of subgroups, there exists an ordinal ζ such that $Z_\zeta(G) = Z_{\zeta+1}(G)$ and consequently $G = Z_\zeta(G)$.

2. In this section we shall give a characterization of the groups G which satisfy $G = Z_\zeta(G)$ for some ordinal ζ.

LEMMA 2.1. *If $G = Z_\zeta(G)$, and if S is a subgroup of G, then $S = Z_\zeta(S)$.*

For one verifies by complete (transfinite) induction that the crosscut of S and $Z_\nu(G)$ is part of $Z_\nu(S)$.

It is a consequence of Jordan-Hölder's theorem that a finite group G is soluble if, and only if, the series of subgroups $C_i(G)$ ends with 1. Thus the following definition of solubility coincides for finite groups with the classical concept.

The (finite or infinite) group G is termed *soluble* whenever there exists an ascending chain of subgroups B_ν for ordinals ν with $0 \leqq \nu \leqq \beta$ so that

(a) $B_0 = 1$;

(b) *B_ν is a normal subgroup of G*;

(c) *$B_{\nu+1}/B_\nu$ is an abelian group of finite rank*;

(d) *B_ν is for limit-ordinals ν the set of all the elements contained in groups B_μ for $\mu < \nu$*;

(e) $B_\beta = G$.

If $G \neq 1$, then there exists a first $B_\nu \neq 1$, so that we may assume without loss in generality that $B_1 \neq 1$.

THEOREM 2.2. *There exists an ordinal ζ so that $G = Z_\zeta(G)$ if, and only if, G is a soluble group, satisfying condition* (G).

Proof. We assume first the existence of an ordinal ζ so that $G = Z_\zeta(G)$. Since every $Z_{\nu+1}(G)/Z_\nu(G)$ is abelian, there exists an ascending chain of subgroups $B_{\nu,\sigma}$ so that $Z_\nu(G) = B_{\nu,0} \leqq B_{\nu,\rho} \leqq B_{\nu,\sigma} \leqq B_{\nu,\tau} = Z_{\nu+1}(G)$ for $\rho < \sigma$, and $B_{\nu,\rho+1}/B_{\nu,\rho}$ is a cyclic group, and so that condition (d) is satisfied by the $B_{\nu,\sigma}$. Since every subgroup of the central of a group is a normal subgroup, it follows that all the $B_{\nu,\sigma}$ are normal subgroups of G; and this proves the solubility of G.

Suppose now that S is a subgroup of G and that T is a greatest subgroup of S. Since $Z_0(S) = 1$, and since T contains $Z_\nu(S)$ whenever ν is a limit-ordinal and T contains all the $Z_\mu(S)$ for $\mu < \nu$, there exists a greatest subscript τ so that $Z_\tau(S) \leqq T$; and since $T < S$ and $S = Z_\zeta(S)$ by Lemma 2.1, it follows that $Z_\tau(S) < S$. Hence there exists an element w in $Z_{\tau+1}(S)$ that is not contained in T. If s is any element in S, then $wsw^{-1}s^{-1}$ is an element in $Z_\tau(S)$ and therefore in T, so that $wT = Tw$. Since T is a greatest subgroup of S, S is generated by

adjoining w to T; and this shows that T is a normal subgroup of G, so that (G) is satisfied by the group G.

Assume now conversely that G is a soluble group and (G) is satisfied by G. Let the B_ν form an ascending chain of subgroups of G which satisfies conditions (a) to (e) and in addition $B_1 \neq 1$. B_1 is a normal abelian subgroup of G. Suppose that K is a subgroup not equal to 1 of B_1 and g an element in G so that $gK = Kg$. Then it follows from (G) and Lemma 1.9 that there exist elements $v \neq 1$ in K so that $gv = vg$. Hence it follows from Lemma 1.10 that $Z(G) \neq 1$.

If G is a soluble group satisfying (G), then every quotient group of G is a soluble group satisfying (G), as may be seen by applying, to the ascending chain B_ν, the homomorphism which defines the quotient group. Since soluble groups not equal to 1, satisfying (G), have a central different from 1, it follows from Lemma 1.11 that there exists an ordinal ζ so that $G = Z_\zeta(G)$.

COROLLARY 2.3. *If G is a finite group, then each of the following properties implies the others:*

(1) *There exists an integer h so that $G = Z_h(G)$.*

(2) *G is a direct product of p-groups.*

(3) *Condition* (G) *is satisfied by the group G.*

The equivalence of (1) and (2) is a well known theorem.* That (3) is a consequence of (1), follows from Theorem 2.2. If finally (3) is satisfied, then it follows from the finiteness of G and from condition (G) that G is soluble by Lemma 1.4; and it follows from Theorem 2.2 that (1) is satisfied by G.†

COROLLARY 2.4. *If* (G) *is satisfied by the group G, and if there exists a finite ascending chain of subgroups B_i so that*

(i) $B_0 = 1$,

(ii) *B_{i-1} is a normal subgroup of B_i and B_i/B_{i-1} is an abelian group of finite rank,*

(iii) $B_k = G$,

then $G = Z_\zeta(G)$ for some (finite or transfinite) ordinal ζ, and each quotient group $Z_{\nu+1}(G)/Z_\nu(G)$ is an abelian group of finite rank.

* Cf. for example, Burnside, *Theory of Groups of Finite Order*.

† It is known (cf. for example H. Wielandt, Mathematische Zeitschrift, vol. 41 (1936), pp. 281–282, or H. Zassenhaus, *Lehrbuch der Gruppentheorie*, vol. 1, p. 108) that for finite groups condition (G) is a consequence of the weaker condition that every greatest subgroup of the whole group be a normal subgroup. Such a condition would be fairly ineffective in the case of infinite groups, since infinite groups need not possess any greatest proper subgroups.

A fact that amounts essentially to the equivalence of (1) and (3) has been proved recently by H. Wielandt, Mathematische Zeitschrift, vol. 45 (1939), pp. 209–244, (16). His condition that every subgroup be "subnormal" (nachinvariant) implies (G) and is equivalent to (G) only in case a suitable chain condition is satisfied; cf. also Theorem 4.13 and Corollary 4.15 below.

Proof. We show by complete induction with regard to i that each of the subgroups B_i fulfills the assertions of our statement. This is clearly true for B_1, since B_1 is an abelian group of finite rank. Thus we may assume that it holds true for B_{i-1}. Since B_{i-1} is a normal subgroup of B_i, and since the $Z_\nu(B_{i-1})$ are characteristic subgroups of B_{i-1}, they are normal subgroups of B_i. Since $Z_{\nu+1}(B_{i-1})/Z_\nu(B_{i-1})$ and $B_i/B_{i-1}=B_i/Z_\rho(B_{i-1})$ are abelian groups of finite rank—by the induction hypothesis—this shows that B_i is a soluble group which satisfies (G); and it follows from Theorem 2.2 that $B_i=Z_\sigma(B_i)$ for some ordinal σ. The subgroup of those classes of $Z_{\nu+1}(B_i)/Z_\nu(B_i)$ which are represented by elements in B_{i-1} is isomorphic with a subgroup of $Z_{\nu+1}(B_{i-1})/Z_\nu(B_{i-1})$ and is therefore an abelian group of finite rank; and $Z_{\nu+1}(B_i)/Z_\nu(B_i)$ is modulo this subgroup isomorphic to a subgroup of B_i/B_{i-1} —an abelian group of finite rank. Thus $Z_{\nu+1}(B_i)/Z_\nu(B_i)$ is an abelian group of finite rank; and this completes the proof.

That it is impossible to prove $G=Z_h(G)$ for integral h may be seen from the following:

EXAMPLE 2.5. *Let B be an abelian group of type p^∞, that is, B is generated by a sequence of elements $b(i)$ so that $b(1)$ is an element of order p, $b(i)^p=b(i-1)$. An automorphism γ of B is defined by $x^\gamma=x^{1+p}$.*

Let G be the group which is generated by adjoining to B an element u, subject to the relation $u^{-1}xu=x^\gamma$ for x in B.

B is the commutator subgroup of G, and G/B is an infinite cyclic group. If i is a positive integer, then $Z_i(G)$ is generated by $b(i)$, so that $B=Z_\omega(G)$, $G=Z_{\omega+1}(G)$. Thus it follows from Theorem 2.2 that (G) is satisfied by G. Finally it may be of interest* to note that $B=C(G)=C^2(G)$.

LEMMA 2.6. *Suppose the subgroups W_ν for $0\leqq\nu\leqq\tau$ of the group G satisfy:*

(a) $W_0=1$;

(b) $W_\nu\leqq W_\mu$ *for* $\nu<\mu$;

(c) W_ν *is for limit-ordinals* ν *the set of all the elements contained in* W_μ *for* $\mu<\nu$;

(d) W_ν *is a normal subgroup of* G;

(e) *if the subgroup S of G contains W_ν, and if w is an element in $W_{\nu+1}$, then $wS=Sw$.*

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* For one proves easily, using an argument of Philip Hall, that if $G=Z_h(G)$ for some integer h, then $C^h(G)=1$; and if $C^k(G)=1$ for some integer k, then $G=Z_k(G)$. That neither part of this statement holds true when infinite indices are admitted, may be seen from the following example that has been discussed in the proof of (1.8). Let G be the group generated by elements $b(i)$ for integral i (positive or negative or zero) and by an element g, subject to the relations $b(i)b(j)=b(j)b(i)$, $b(i)^2=1$, $g^{-1}b(i)g=b(i+1)$. $C(G)$ is abelian, $Z(G)=1$, (G) is not satisfied by G, but the crosscut of the groups $C^i(G)$ for positive integers i consists of the identity only.

Then (G) *is satisfied by the group G, if (and only if)* (G) *is satisfied by the quotient group* G/W_τ.

Proof. Suppose that S is a subgroup of G, and that T is a greatest subgroup of S. Denote by S_ν the crosscut of S and W_ν, and by T_ν the crosscut of T and W_ν. If ν is a limit-ordinal, and if $S_\mu = T_\mu$ for every $\mu < \nu$, then $S_\nu = T_\nu$ by (c). Since $S_0 = T_0 = 1$, there exists therefore a greatest ordinal ρ so that $S_\rho = T_\rho$. If $\rho = \tau$, then T/T_τ is a greatest subgroup of S/S_τ, and S/S_τ is essentially a subgroup of G/W_τ—a group which satisfies (G)—so that T is a normal subgroup of S. If $\rho < \tau$, then there exists an element w in $S_{\rho+1}$ which is not contained in T. It is a consequence of (e) that there exists for every element g in G an integer i and an element v in W_ρ so that $w^{-1}gw = g^i v$. If g is in particular an element in S, then v is an element in S, and therefore an element in $S_\rho = T_\rho \leqq T$. Thus $wT = Tw$. Since w is not in T, S is generated by adjoining w to T, and hence T is a normal subgroup of S.

The *norm* $N(G)$ of the group G has been defined* as the set of all the elements g in G which satisfy $gS = Sg$ for every subgroup S of G. The iterated norms $N_\nu(G)$ may be defined as usual. An obvious consequence of Lemma 2.6 is the fact that (G) is satisfied by a group G, whenever $G = N_\kappa(G)$ for some ordinal κ. Using Corollary 2.3 this implies the following statement:

If G is a finite group, then properties (1) *to* (3) *of Corollary* 2.3 *are equivalent to the following condition*:

(4) $G = N_\tau(G)$ *for some integer* r.

If S is a greatest abelian subgroup of the group G, then it is clear that S equals its centralizer in G. It is furthermore known† that every greatest normal and abelian subgroup of G equals its centralizer in G, if only $G = Z_h(G)$ for some integer h. This may be generalized as follows.

If S is a greatest normal and abelian subgroup of the group G, and if $G = Z_\zeta(G)$ *for some (finite or infinite) ordinal* ζ, *then the centralizer of S in G is equal to S.*

Proof. Denote by T the centralizer of S in the group G. Since S is normal and abelian, $S \leqq T$ and T is a normal subgroup of G. Denote by S_ν the crosscut of S and $Z_\nu(G)$; and denote by T_ν the crosscut of T and $Z_\nu(G)$. Since $S_0 = T_0 = Z_0(G) = 1$, we may assume that $S_\mu = T_\mu$ has been proved for every $\mu < \nu$.

CASE 1. $\nu = \rho + 1$.

Then $S_\rho = T_\rho$. If u is any element in T_ν, s an element in G, then $usu^{-1}s^{-1}$

* Cf. Compositio Mathematica, vol. 1 (1934), pp. 254–283.

† Cf. H. Zassenhaus, *Lehrbuch der Gruppentheorie*, vol. 1, p. 108.

is an element in $Z_\rho(G)$ as well as in T; and therefore it is an element in T_ρ. Hence it is an element in S. Thus the subgroup U, generated in adjoining u to S, is a normal subgroup of G. Since u is an element in T, U is abelian. Hence $U=S$ so that $T_\nu \leqq S$ or $S_\nu = T_\nu$.

CASE 2. ν is a limit-ordinal.

Then S_ν is the join of the S_μ for $\mu<\nu$; and T_ν is the join of the T_μ for $\mu<\nu$; and this implies that $S_\nu = T_\nu$.

Thus it follows in particular that $S=S_\xi=T_\xi=T$, since $G=Z_\xi(G)$.

3. In this section we are going to apply the results of the preceding sections on groups without elements of infinite order; in particular we shall be interested in the possibility of decomposing a group into a direct product of p-groups.

3.A. For some of our purposes it will be sufficient to assume instead of solubility a somewhat weaker property. A group G may be termed *metacyclic*, if there exists an ascending chain of subgroups B_ν with the properties:

(a) $B_0=1$;

(b) *B_ν is a normal subgroup of $B_{\nu+1}$ and $B_{\nu+1}/B_\nu$ is abelian*;

(c) *B_ν is for limit-ordinals ν the set of all the elements which are contained in groups B_μ for $\mu<\nu$*;

(d) $B_\beta=G$.

If $G\neq 1$, then there exists a first $B_\nu\neq 1$, so that we may assume without loss in generality that $B_1\neq 1$. One verifies readily that every abelian group possesses an ascending chain of subgroups B_ν satisfying (a), (c), (d) and instead of (b) the stronger condition that $B_{\nu+1}/B_\nu$ be cyclic. This shows that there is no loss of generality in assuming that the chain B_ν which defines G as a metacyclic group satisfies (a), (c), (d) and

(b′′) *B_ν is a normal subgroup of $B_{\nu+1}$ and $B_{\nu+1}/B_\nu$ is cyclic.*

If there exists a chain of subgroups B_ν satisfying (a), (b′′), (c) and (d), then G is said to be *metacyclic of length less than or equal to β.*

Soluble groups are metacyclic; and these two concepts coincide for finite groups. But it is easy to construct examples of infinite metacyclic p-groups which are not soluble, though they satisfy condition (G); cf. Example 3.4 below.

LEMMA 3.1. *Every metacyclic group without elements of infinite order which is generated by a finite number of elements is finite.*

Proof. If G is metacyclic, then there exists an ascending chain of subgroups B_ν which satisfies the above conditions (a) to (d). We may assume without loss of generality that $B_\mu<B_\nu$ for $\mu<\nu$. If the length of the chain

is 1, then the group is abelian; and abelian groups without elements of infinite order are finite, if they are generated by a finite number of elements. Hence we may assume that our statement holds true for metacyclic groups possessing a chain of shorter length. Since G is generated by a finite number of elements, it is impossible that the length β be a limit-ordinal, since otherwise there would exist an ordinal $\gamma<\beta$ so that all the elements in the finite set of generators of G are contained in B_γ and so that therefore $B_\beta=G=B_\gamma$ for $\gamma<\beta$ which contradicts our hypothesis concerning the chain B_ν. Hence $\beta=\alpha+1$ and B_α is a normal subgroup of G. Since G/B_α is an abelian group without elements of infinite order which is generated by a finite number of elements, it is a finite group; and it follows from (1.3) that B_α is generated by a finite number of elements. It is therefore a consequence of the induction hypothesis that B_α is finite; and G is finite, since its quotient group, modulo a finite normal subgroup, is finite.

THEOREM 3.2. *Each of the following three properties of a group G without elements of infinite order implies the others.*

(1) *$G=Z_\zeta(G)$ for some ordinal ζ.*

(2) *G is a direct product of soluble p-groups.*

(3) *G is soluble and satisfies condition* (G).

REMARK. It may be verified easily that all the elements of a soluble group are of finite order if, and only if, there exists an ascending chain of subgroups B_ν satisfying conditions (a) to (e) of §2 and in addition:

(c′) *$B_{\nu+1}/B_\nu$ is a finite abelian group.*

Proof. The equivalence of (1) and (3) has been established as Theorem 2.2. In order to prove the equivalence of (1) and (2) we show first

(3.2.1) *If G satisfies* (1), *and if the subgroup S of G is generated by a finite number of elements, then S is a finite group and a direct product of p-groups.*

It is a consequence of Lemma 2.1 that there exists an ordinal σ so that $S=Z_\sigma(S)$; and it is a consequence of Theorem 2.2 that S is soluble and metacyclic. Hence it follows from Lemma 3.1 that S is finite; and it is a consequence of Corollary 2.3 that S is a direct product of p-groups.

Denote now by u and v two elements of order a power of the same prime number p. The subgroup S, generated by u and v, is by (3.2.1) the direct product of its primary components and is consequently a p-group. Hence uv is an element of order a power of p. This shows in particular that the set G_p of all the elements in G whose order is a power of p is a subgroup of G; and one verifies now as usual that G is the direct product of its primary components G_p.

Since G_p is a subgroup of G, there exists by Lemma 2.1 some ordinal α so that $G_p = Z_\alpha(G_p)$. This implies by Theorem 2.2 that G_p is soluble. Thus (2) is a consequence of (1).

In order to prove that (1) is a consequence of (2) we show first

(3.2.2) *If G is a soluble p-group, then there exists an ordinal ζ so that $G = Z_\zeta(G)$.*

Since $G \neq 1$ is a soluble p-group, there exists an ascending chain of subgroups B_ν which satisfies conditions (a) to (e) of §2 and in addition the condition that B_1 is a finite abelian p-group not equal to 1.

Suppose now that $K \neq 1$ is a subgroup of B_1, and that g is an element in G so that $gK = Kg$. Then g induces in K an automorphism whose order is a power of p; and one proves as usual* that such an automorphism of K possesses fixed-elements not equal to 1; that is, there exist elements $v \neq 1$ in K so that $gv = vg$. Hence the central of a soluble p-group is different from 1 whenever the group is different from 1, as follows from Lemma 1.10.

Quotient groups of soluble p-groups are soluble p-groups. Hence (3.2.2) is a consequence of Lemma 1.11 and the fact just proved.

Suppose now that (2) is satisfied by (G). Then G is the direct product of soluble p-groups H_ν. To every H_ν there exists by (3.2.2) an ordinal $\zeta(\nu)$ so that $H_\nu = Z_{\zeta(\nu)}(H_\nu)$. Let ζ be some ordinal which is greater than all the $\zeta(\nu)$. Since it may be proved by complete (transfinite) induction that $Z_\rho(G)$ is the direct product of the groups $Z_\rho(H_\nu)$, it follows in particular that $G = Z_\zeta(G)$ so that (1) is a consequence of (2).

The theorem that nilpotent groups without elements of infinite order are direct products of their primary components has been proved by deriving (3.2.1) and by applying Corollary 2.3, that is, by reduction to the finite case. An *alternative proof* may be offered which does not make use of this reduction and which consequently includes a proof of the theorem for finite groups too.

We have verified before that groups G satisfying $G = Z_\zeta(G)$ are metacyclic so that the absence of elements of infinite order implies the existence of an ascending chain of subgroups B_ν with the following properties:

(a) $B_0 = 1$;

(b) B_ν is a normal subgroup of $B_{\nu+1}$;

(c) $B_{\nu+1}/B_\nu$ is a cyclic group of order a prime number;

(d) B_ν is for limit-ordinals ν the set of all the elements which are contained in groups B_μ for $\mu < \nu$;

(e) $B_\beta = G$.

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* The elements $g^{-i}xg^i$ for integral i form a set containing $p^{n(x)}$ elements. Since $n(1) = 0$, there exists at least one element $y \neq 1$ such that $n(y) = 0$, since the number of elements in K is a power of p.

Since B_0 is the direct product of its primary components, we may assume that it has already been proved that every B_μ for $\mu<\nu$ is the direct product of its primary components $B_{p\mu}$.

CASE 1. $\nu=\rho+1$.

The index of B_ρ in B_ν is a certain prime number p; and B_ρ is the direct product of its primary components $B_{q\rho}$ so that we may represent B_ρ as the direct product of $B_{p\rho}$ and of the direct product H of all the $B_{q\rho}$ for q not equal to p.

Let u be any element generating B_ν modulo B_ρ. The order of u has the form ip^j where i is relatively prime to p and $0<j$. Then u^i generates B_ν modulo B_ρ, so that there exists an element v of order a power of p which generates B_ν modulo B_ρ.

If h is an element in H, then we are going to prove that $hv=vh$. This is certainly true if h is an element in $Z_0(G)$. Hence we may assume that this has been proved for all those elements in H which are contained in groups $Z_\lambda(G)$ for $\lambda<\sigma$. If h is an element in the crosscut of H and $Z_\sigma(G)$ which is not contained in any $Z_\lambda(G)$ for $\lambda<\sigma$, then $c=vhv^{-1}h^{-1}$ is an element in some $Z_\kappa(G)$ for $\kappa<\sigma$. Since H is a characteristic subgroup of B_ρ, and since B_ρ is a normal subgroup of B_ν, H is a normal subgroup of B_ν and c is an element in H. Hence it follows from the induction hypothesis that $cv=vc$, and consequently we find that $v^thv^{-t}=c^th$ for integral t. If t is in particular the order of v, then $c^t=1$. Since the order of v is a power of p, and since the orders of the elements in H are all relatively prime to p, it follows that $c=1$ or $vh=hv$. Since $G=Z_f(G)$, this implies that $hv=vh$ for every h in H.

If x is any element in $B_{p\rho}$, i a positive integer, then

$$(xv^i)^p = xv^ixv^{-i}v^{2i}xv^{-2i}\cdots v^{(p-1)i}xv^{-(p-1)i}v^{pi},$$

and all the factors in the expression on the right are elements in $B_{p\rho}$, so that $(xv^i)^p$ and therefore xv^i itself is an element of order a power of p. The group K, generated by adjoining v to $B_{p\rho}$, is therefore a p-group.

If f is an element in K, h an element in H, then $fh=hf$, since this holds true both for $f=v$ and for f in $B_{p\rho}$. This shows that K contains all the elements of order a power of p in B_ν, and that H contains exactly those elements in B_ν whose orders are prime to p, since B_ν is generated by the elements in K and in H. Hence B_ν is the direct product of K and H, that is, B_ν is the direct product of its primary components.

Case 2. ν is a limit-ordinal.

If u and v are two elements in B_ν whose orders are powers of the same prime number p, then the order of uv is a power of p too, since both u and v are contained in some B_μ for $\mu<\nu$. The elements of order a power of p in B_ν

form therefore a subgroup of B_ν; and now it follows that B_ν is the direct product of its primary components.

Accordingly every B_ν is the direct product of its primary components; and this holds true in particular for $B_\beta = G$.

This proof shows slightly more than we intended to prove. For our argument contains a proof of the following statement.

COROLLARY 3.3. *If all the elements in G are of finite order, if $uv = vu$ for elements u and v in G whose orders are relatively prime, and if G is metacyclic, then G is the direct product of its primary components.*

REMARK. If G is a finite group of order $\prod_p p^{n(p)}$, then it contains a Sylow subgroup S_p of order $p^{n(p)}$; and G is generated by any such system of Sylow representatives. If G satisfies the condition

(5) *$uv = vu$, if the orders of u and v are relatively prime,*

then the subgroup generated by a system of Sylow representatives is their direct product. This shows that *a finite group is a direct product of p-groups if, and only if, it satisfies condition* (5).

It should however be noted that for the proof of this fact we needed the existence of Sylow subgroups, a fact that is comparatively much deeper than the fairly elementary means employed in the proof of Corollary 3.3.

EXAMPLE 3.4. *There exist infinite p-groups whose central is* 1, *though their commutator subgroup is abelian.*

This example shows in particular that it is impossible to omit the solubility in (2) of Theorem 3.2.

Let B be a countable abelian group all of whose elements not equal to 1 are of order the prime number p, so that there exists a basis $b(1), b(2), \cdots, b(i-1), b(i), \cdots$ of B.

As is well known, there exists for every non-negative integer a unique p-adic expansion $\sum_{i=0}^{\infty} c_i p^i$ where $0 \leqq c_i < p$ and where all the c_i—apart from a finite number of exceptions—are 0.

An automorphism $\gamma(j)$ for $0 < j$ of B is defined by

$$b\left(\sum_{i=0}^{\infty} c_i p^i\right)^{\gamma(j)} = b\left(\sum_{i=0}^{\infty} c_{ij} p^i\right)$$

for

$$\begin{aligned} c_{ij} &= c_i, && \text{if} && i \neq j-1, \\ c_{j-1\,j} &= c_{j-1} + 1 && \text{for} && c_{j-1} < p-1, \\ &= 0 && \text{for} && c_{j-1} = p-1. \end{aligned}$$

These automorphisms γ are all of order p and generate a commutative group of automorphisms. Hence there exists a group G which is generated by adjoining to B elements $u(j)$, subject to the relations

$$u(j)^p = 1,\ u(j)u(h) = u(h)u(j), \qquad b(i)u(j) = u(j)b(i)^{\gamma(j)}.$$

B is the commutator subgroup of this group G; and all the elements not equal to 1 in G are of order p.

If U is the subgroup generated by the elements $u(j)$, then U is an abelian subgroup of G, the crosscut of B and U is 1, and every element in G may be represented uniquely in the form bu for b in B and u in U.

If $u \neq 1$, then u does not permute with $b(1)$. If $b \neq 1$, then there exists an integer i so that b is a product of elements $b(j)$ for $0 \leqq j < p^i$; and hence $b \neq b^{\gamma(i+1)}$, so that b does not permute with $u(i+1)$. Thus the central of G is 1.

If b is an element in B, u in U, then $buU = Ubu$ if, and only if, $bU = Ub$. If $b \neq 1$, then there exists an element v in U—as has been pointed out just now—so that $vb \neq bv$. Since the commutator of b and of v is an element not equal to 1 in B, this shows that $bU = Ub$ implies $b = 1$; and we have proved that *the normalizer of U in G is just U.*

Any finite number of elements in G generates a finite subgroup of G. It is a consequence of Theorem 4.1 below that this implies that (G) is satisfied by G, since finite p-groups satisfy (G).

If the subgroup S of G is generated by adjoining a finite number of elements $u(i)$ to B, then S is a normal subgroup of G, so that $S = Z_\sigma(S)$ for some ordinal σ. But the join of these subgroups is G, and $Z(G) = 1$.

3.B. A characteristic property of the groups without elements of infinite order which satisfy condition (G) is given in the following theorem:

THEOREM 3.5. *Suppose that G be a group without elements of infinite order; and denote by $W(G)$ the crosscut of $Z(G)$ and of the iterated commutator subgroups $C_i(G)$. Then condition* (G) *is satisfied by G if, and only if, $G/W(G)$ is the direct product of its primary components, and if these primary components satisfy* (G).

REMARK. It is still an open question whether or not all p-groups satisfy (G), and whether or not there exist groups G without elements of infinite order which satisfy (G), though they are not direct products of p-groups.

Proof. We assume first that (G) is satisfied by the group G; and we derive from this hypothesis that

(3.5.1) *$uv = vu$, if the orders of u and v are relatively prime.*

For, denote by U the subgroup generated by u and v. Since all the elements in G are of finite order, and since (G) is satisfied by G and therefore

by U, it follows from Lemma 1.4 that all the quotient groups $U/C_i(U)$ are finite, and that either $C_i(U)=1$ or $C_{i+1}(U)<C_i(U)$. Hence it follows from Corollary 2.3 that every $U/C_i(U)$ is a direct product of p-groups, so that in particular: $uv \equiv vu$ modulo $C_i(U)$, since the orders of u and v are relatively prime. Hence every $U/C_i(U)$ is abelian, so that $C(U)=C_2(U)$ and consequently $C(U)=1$. Hence U is abelian, and this proves our statement.

Denote now by S the subgroup generated by two elements x and y both of order a power of the same prime number p. The order of xy is a positive number ip^j for $0 \leqq j$ and i relatively prime to p. Put $z=(xy)^{p^j}$. Then z is an element of order i. If s is an element of order a power of p, then $sz=zs$ by (3.5.1). If the order of s is relatively prime to p, then it follows from (3.5.1) that $sx=xs$ and $sy=ys$; and this implies $sz=zs$, so that z is an element in $Z(G)$.

It is a consequence of Lemma 1.4 that $S/C_k(S)$ is a finite group. Since this quotient group is generated by two elements of order a power of p, and since it satisfies (G), it follows from Corollary 2.3 that $S/C_k(S)$ is itself a p-group. Hence z is an element in $C_k(S) \leqq C_k(G)$ for every k so that z is an element in $W(G)$. The element xy is therefore modulo $W(G)$ an element of order a power of p; and this proves that $G/W(G)$ is a direct product of p-groups each of which satisfies (G).

Suppose now that the group H is the direct product of its primary components H_p, and that (G) is satisfied by each H_p. Every subgroup S of H is the direct product of its primary components S_p, since S_p is the crosscut of S and H_p. If T is a greatest subgroup of S, then there exists a prime number q so that the primary component $T_q<S_q$. If $p \neq q$, then the join of T and S_p is a subgroup between T and S which is different from S, since it contains only a proper subgroup of S_q, as T is the direct product of its primary components. Hence $S_p \leqq T$ or $S_p=T_p$. If $T_q<K \leqq S_q$, then the join of T and K is a subgroup between T and S which is different from T and which is therefore equal to S. Hence $K=S_q$ so that T_q is a greatest subgroup of S_q. Since S_q satisfies (G), it follows that T_q is a normal subgroup of S_q, so that T is a normal subgroup of S.

Assume now that $G/W(G)$ is the direct product of its primary components, and that these primary components satisfy (G). Then we have proved just now that $G/W(G)$ satisfies (G), and it follows from Lemma 2.6 that G satisfies (G), since $W(G) \leqq Z(G)$.

3.C. If a group G contains both elements of finite and of infinite order, then it may happen that the elements of finite order do not form a subgroup (example: the group generated by two elements u and v, subject to the rela-

tions $u^2=v^2=1$, where uv is of infinite order). Thus it is of interest to find criteria excluding this possibility.

LEMMA 3.6. *If* (G) *is satisfied by the group G, if G is generated by a finite number of elements each of which is of finite order, if there exists a normal subgroup H of G so that H is abelian and so that G/H is a finite group, then G is a finite group.*

Proof. Our statement is certainly true if the index of H in G is 1, since in this case $G=H$, that is, G is an abelian group generated by a finite number of elements each of which is of finite order. Thus we may assume that our statement is true for all the pairs G', H' which satisfy the hypothesis of the lemma, provided the index of H' in G' is smaller than the index of H in G; and we may assume $G\neq H$.

Suppose furthermore that K is a subgroup of H and that K is a normal subgroup of G. Then G/H and $[G/K]/[H/K]$ are essentially the same groups; and all the hypotheses of the lemma concerning G and H are satisfied by G/K and H/K.

It will be convenient to put $G/K=G^*$ and $H/K=H^*$, though the conclusions we are going to arrive at now will later be applied to different subgroups K.

Since G^* is generated by a finite number of elements, and since G^*/H^* is a finite group, it follows from (1.3) that H^* *is generated by a finite number of elements.*

Since G^* satisfies condition (G), it follows from Lemma 1.9 that there exists for every element g in G^* and for every subgroup R of H^*, satisfying $gR=Rg$ and $R\neq 1$, an element $r\neq 1$ in R so that $gr=rg$. Since G^*/H^* is a finite group satisfying (G), it follows from Corollary 2.3 that $G^*/H^*=Z_k(G^*/H^*)$ for some positive integer k; and consequently it follows from Lemma 1.10 that *the crosscut of* $Z(G^*)$ *and* H^* *is different from* 1, *if* $H^*\neq 1$.

Since H is an abelian group, the elements of finite order in H form a subgroup F of H. Our lemma will be proved as soon as we know that $H=F$, since H is generated by a finite number of elements. Thus let us suppose that $F<H$. If K is a subgroup of H so that $H/K\neq 1$ and does not contain elements of finite order not equal to 1, and so that K is a normal subgroup of G, then $F\leqq K$; and the crosscut K'' of the central of G/K and of H/K is different from 1, and modulo K'' there are no elements not 1 of finite order in H/K. If K' is the subgroup between K and H so that $K''=K'/K$, then the rank of H/K' is smaller than the rank of H/K, since both ranks are finite; and the same hypotheses that applied to K apply to K'. Thus we find after a finite number of steps that $F\neq H$ implies the existence of a subgroup N of H with

the following properties: *N is a normal subgroup of G; H/N does not contain any elements not equal to* 1 *of finite order; $N<H$; H/N is part of the central of G/N.*

Denote by Z the subgroup between H and G so that Z/H is the central of G/H. Since G/H is a finite group—which we may assume now to be different from 1—and since G/H satisfies (G), it follows from Corollary 2.3 that $Z/H\neq 1$. The group H/N is part of the central of G/N and is therefore part of the central of Z/N. Since Z/H and $[Z/N]/[H/N]$ are essentially the same, and since the first of these groups is abelian, it follows that Z/N is a group with abelian central quotient group. Since Z/H is a finite abelian group, it follows from the well known properties of groups with abelian central quotient group that all the commutators of elements in Z/N are elements of finite order which are contained in H/N. Since all the elements not 1 in H/N are of infinite order, this proves that *Z/N is an abelian group.*

If $G'=G/N$, $H'=Z/N$, then (G) is satisfied by G', G' is generated by a finite number of elements each of which is of finite order, H' is a normal abelian subgroup of G', and G'/H' is a finite group whose order is smaller than the order of G/H. Hence it follows from the induction hypothesis, stated in the first paragraph of the proof, that G' is a finite group; and this implies in particular that $H'=Z/N$ is a finite group. But this is impossible, since H is between Z and N, and since N has been determined in such a fashion that H/N is an infinite group. Thus we find that the assumption $F\neq H$ leads to a contradiction, that is, $F=H$ is a finite abelian group; and G is a finite group.*

COROLLARY 3.7. *If* (G) *is satisfied by the group G, if G is generated by a finite number of elements each of which is of finite order, then $G/C_i(G)$ is a finite group, $C_i(G)$ is generated by a finite number of elements; and $C_{i-1}(G)\neq 1$ implies $C_i(G)<C_{i-1}(G)$.*

* The classical theorem that a finite group G is a p-group if it satisfies $G=Z_k(G)$ and is generated by elements whose orders are powers of p, may be proved by the same method as follows.

Since the theorem certainly holds true for groups of order 1, we may assume that it holds true for all quotient groups of G whose orders are smaller than the order of G. Thus in particular $G/Z(G)$ is a p-group. If $Z(G)$ itself is a p-group, then our theorem has been proved for G. Otherwise $Z(G)$ is the direct product of two groups P and Q where all the elements in P are of order a power of p, and where the orders of the elements in Q are prime to p. P is a normal subgroup of G, since it is a characteristic subgroup of $Z(G)$. If $P\neq 1$, then the order of G/P is smaller than the order of G, so that G/P would be a p-group and $Q=1$, that is, G itself would be a p-group. Thus assume that $P=1$. Then $Q=Z(G)$ is part of the central of $Z_2(G)=R$, and R/Q is an abelian p-group. Now it follows from the well known properties of groups with abelian central quotient group that the commutators of elements in R are elements in Q whose orders are powers of p. Hence they are all equal to 1, and $Z_2(G)=R$ is an abelian group. As R/Q is a p-group, R is the direct product of Q and a p-group S. If $S=1$, then $Z(G)=Z_2(G)$ and G is an abelian group and as such a p-group. If $S\neq 1$, then G/S is of lower order than G and is therefore a p-group. Since Q is isomorphic with a subgroup of G/S, it follows that $Q=1$ so that G is a p-group in every case.

REMARK. This statement is an improvement upon Lemma 1.4.

Proof. Since our statement is certainly true for $i=0$, we may assume that it has been verified already for $i-1$. Thus $G/C_{i-1}(G)$ is a finite group, $C_{i-1}(G)$ is generated by a finite number of elements, and $C_{i-1}(G)\neq 1$ implies $C_i(G)<C_{i-1}(G)$. If $G'=G/C_i(G)$, $H'=C_{i-1}(G)/C_i(G)$, then G'/H' and $G/C_{i-1}(G)$ are essentially the same finite groups, so that Lemma 3.6 may be applied to G', H'. Hence G' is a finite group. Since therefore $G/C_i(G)$ is a finite group, and since G is generated by a finite number of elements, it follows from (1.3) that $C_i(G)$ is generated by a finite number of elements; and it follows from (1.2) that either $C_i(G)=1$ or $C_{i+1}(G)<C_i(G)$.

COROLLARY 3.8. *If condition* (G) *is satisfied by the group G, and if u, v are elements of relatively prime finite orders in G, then $uv=vu$.*

REMARK. This statement improves upon (3.5.1).

Proof. If W is the group generated by u and v, then it is a consequence of Corollary 3.7 that $W/C_2(W)$ is a finite group. Since $W/C_2(W)$ satisfies (G), and since the orders of u and v are relatively prime, it follows from Corollary 2.3 that $uv\equiv vu$ modulo $C_2(W)$, so that $W/C_2(W)$ is abelian, since this quotient group is generated by u and v. Hence $C(W)=C_2(W)$; and it follows from Corollary 3.7 that $C(W)=C_2(W)=1$, so that W is abelian and in particular $uv=vu$.

3. D. For the following considerations it will be convenient to have a concept intermediate between "metacyclic" and "soluble." A group G shall be termed *weakly soluble* if there exists an ascending chain of subgroups B_ν with the following properties:

(a) $B_0=1$;

(b) *B_ν is a normal subgroup of G*;

(c) *$B_{\nu+1}/B_\nu$ is abelian*;

(d) *B_ν is for limit-ordinals ν the set of all the elements contained in subgroups B_μ for $\mu<\nu$*;

(e) $B_\tau=G$.

One verifies that soluble groups are weakly soluble, and that weakly soluble groups are metacyclic. If $G\neq 1$, then there is no loss of generality in assuming that $B_\mu<B_\nu$ for $\mu<\nu$.

THEOREM 3.9. *If* (G) *is satisfied by the weakly soluble group G, and if G is generated by a finite number of elements each of which is of finite order, then G is finite.*

REMARK. Finite groups, satisfying (G), are, by Corollary 2.3, soluble groups.

Proof. Since G is weakly soluble, there exists a properly ascending chain of subgroups B_ν, satisfying (a) to (e). The ordinal τ may be represented uniquely in the form $\tau=\sigma+n$ where σ is either 0 or a limit-ordinal, and where n is a non-negative integer. We are now going to prove that $G/B_{\tau-i}$ is a finite group for $0\leqq i\leqq n$. Since this is certainly true for $i=0$, we may suppose that it holds true for $i-1$. If we put $J=G/B_{\tau-i}$, $H=B_{\tau-i+1}/B_{\tau-i}$, then H is abelian, $J/H=G/B_{\tau-i+1}$ is finite, and J is generated by a finite number of elements each of which is of finite order. Hence it follows from Lemma 3.6 that J itself is finite.

Thus in particular G/B_σ is a finite group. Since G is generated by a finite number of elements, it follows from (1.3) that B_σ is generated by a finite number of elements. If σ were a limit-ordinal, then this would imply $B_\sigma=B_\rho$ for some $\rho<\sigma$. This is impossible, since the B_ν form a properly ascending chain of subgroups. Hence $B_\sigma=B_0=1$, so that $G/B_\sigma=G/B_0=G$ is a finite group.

COROLLARY 3.10. *If condition* (G) *is satisfied by the weakly soluble group G, then the subgroup, generated by the elements of finite order in G, is a direct product of p-groups* [*and does not contain elements of infinite order*].

REMARK. It would be sufficient to assume that subgroups of G which are generated by a finite number of elements of finite order satisfy (G) and are weakly soluble. The group generated by two elements u and v, subject to the relations $u^2=v^2=1$, is soluble though uv is an element of infinite order. This shows that condition (G) cannot be omitted. Whether or not the weak solubility is really needed is still an open question.

Proof. If u and v are two elements of finite order, then it follows from the hypothesis and from Theorem 3.9 that they generate a finite group, so that uv is an element of finite order. The elements of finite order in G form therefore a subgroup F without elements of infinite order. It follows furthermore from Corollary 3.8 that $xy=yx$, if x and y are elements in H whose orders are relatively prime. Since G is weakly soluble, F is metacyclic, and it follows now from Corollary 3.3 that F is a direct product of p-groups.

4.A. In this section "properties in the large" will be derived from "properties in the small."

THEOREM 4.1. *Condition* (G) *is a consequence of the following property*:

(G*) *If T is a greatest subgroup of the subgroup S of G, and if S is generated by a finite number of elements, then T is a normal subgroup of S.*

REMARK. This theorem has two important, immediate consequences.

(i) In every group there exists at least one greatest subgroup satisfying condition (G).

(ii) In every group there exists at least one greatest normal subgroup satisfying (G).

Proof. Suppose that S $[<T]$ is a greatest subgroup of the subgroup T of G. Let t be any element in T which is not contained in S, let s be some element in S, and put $c=sts^{-1}t^{-1}$. If c were not contained in S, then T would be generated by adjoining c to S. Hence there would exist a finite number s_i of elements in S so that t is contained in the subgroup V generated by the elements c_1S and s_i. Since the s_i, s are contained in the crosscut U of S and V, and since t is not contained in S, there exists a greatest subgroup W of V which contains $[s, s_i$ and$]$ U, but not t. If $W<M\leqq V$, then t is in M, so that c is in M, since s is in W. Hence M contains s, s_i, c, that is, $M=V$. V is generated by a finite number of elements. W is a greatest subgroup of V. Consequently W is a normal subgroup of V. Since t is in V, and since s is in W, this implies that c is in W, so that t itself belongs to W. This is impossible. Thus it follows that every $tst^{-1}s^{-1}$ for t in T and s in S is an element in S, that is, S is a normal subgroup of T. Hence (G) is satisfied by G.

COROLLARY 4.2. *The group G satisfies* (G) *and has at the same time the property that any finite subset generates a finite subgroup if, and only if, G is the direct product of p-groups with the property that their finite subsets generate finite subgroups.*

REMARK. It is a consequence of this fact that the group constructed as Example 3.4 satisfies condition (G).

Proof. Suppose first that G satisfies (G) and has the property that finite subsets generate finite subgroups. Then all the elements in G are of finite order. Let u and v be elements both of order a power of the prime number p. The subgroup generated by u and v is finite and satisfies (G). Hence it follows from Corollary 2.3 that this subgroup is a finite p-group, so that uv is of order a power of p. Now it follows again that G is the direct product of its primary components, and their finite subsets clearly generate finite subgroups.

Suppose conversely that G is the direct product of its primary components G_p and that finite subsets of G_p, generate finite subgroups of G_p. Let now R be a finite subset of G. Then there exists clearly a finite set S whose elements are each of them of prime power order so that the subgroup T, generated by S, contains all the elements in R. T is easily verified to be the direct product of finite p-groups. R generates therefore a finite group which, by Corollary 2.3, satisfies (G). Now it follows from Theorem 4.1 that G itself satisfies (G), and this completes the proof.

The following statement is an immediate consequence of Lemma 3.1, (3.5.1) and Corollary 4.2.

THEOREM 4.3. *A metacyclic group without elements of infinite order satisfies condition* (G) *if, and only if, it is the direct product of its primary components.*

Combining this result and Corollary 3.10, the following may be proved.

A weakly soluble group is generated by elements of finite order and satisfies condition (G) *if, and only if, it is a direct product of p-groups.*

4.B. It is possible to derive stronger results, if one restricts the length of the ascending subgroup-chains which exhaust the group (cf. Corollary 2.4).

LEMMA 4.4. *Suppose that* (G) *is satisfied by the group G, and that there exists a finite ascending chain of subgroups B_i with the following properties*:

(i) B_1 *is abelian*;

(ii) B_i *is a normal subgroup of* B_{i+1} *and* B_{i+1}/B_i *is a cyclic group*;

(iii) $B_n = G$.

Then $G = Z_\zeta(G)$ *for some suitable ordinal* ζ.

Proof. Suppose that the group G satisfies the hypothesis of the lemma and that $G \neq 1$. Then we may assume that $B_1 \neq 1$.

Clearly $Z(B_1) = B_1 \neq 1$. Hence assume that it has already been proved that $Z(B_i) \neq 1$. Since B_i is a normal subgroup of B_{i+1}, and since $Z(B_i)$ is a characteristic subgroup of B_i, it follows that $Z(B_i)$ is a normal subgroup of B_{i+1}. Since B_{i+1}/B_i is a cyclic group, there exists an element b which generates B_{i+1} modulo B_i. Since $Z(B_i)$ is an abelian group not equal to 1, since $bZ(B_i) = Z(B_i)b$, and since the subgroup generated by b and by $Z(B_i)$ satisfies (G), it follows from Lemma 1.9 that there exists an element $v \neq 1$ in $Z(B_i)$ so that $bv = vb$. Clearly v is an element in $Z(B_{i+1})$, so that this latter subgroup is different from 1. Hence it follows by complete induction that $Z(G) \neq 1$, if $G \neq 1$.

The last result and Lemma 1.11 imply finally that $G = Z_\zeta(G)$.

COROLLARY 4.5. *A group G with abelian commutator subgroup satisfies condition* (G) *if, and only if, its finite subsets generate subgroups S so that* $S = Z_\sigma(S)$ *for suitable ordinals* σ.

Proof. The sufficiency of the condition is a consequence of Theorems 2.2 and 4.1. If conversely (G) is satisfied by G, and if the subgroup S of G is generated by a finite number of elements, then $C(S)$ is abelian and $S/C(S)$ is generated by a finite number of elements. Hence Lemma 4.4 may be applied to S, and this proves the necessity of the condition.

COROLLARY 4.6. *A metacyclic group of length less than or equal to* ω *satisfies condition* (G) *if, and only if, its finite subsets generate subgroups S so that* $S = Z_\sigma(S)$ *for suitable ordinals* σ.

Proof. The sufficiency of the condition is a consequence of Theorems 2.2 and 4.1. If conversely (G) is satisfied by the metacyclic group H of finite length, then it is a consequence of Lemma 4.4 that $H=Z_\eta(H)$ for suitable ordinals η. If G is a metacyclic group of length less than or equal to ω, then the subgroups of G are metacyclic groups of length not exceeding ω, so that finite subsets generate metacyclic groups of finite length; and this proves the necessity of the condition.

4.C. We return now to the analysis of condition (G*) of Theorem 4.1.

THEOREM 4.7. *If the group G is generated by a finite number of elements, then the following two conditions are equivalent:*

(a) *Every greatest subgroup of G is a normal subgroup of G.*

(b) *A set of elements in G generates G if, and only if, it generates G modulo $C(G)$.**

REMARK. Since $G=C(G)$ implies that the identity generates G modulo $C(G)$, it follows that $G\neq 1$, $G=C(G)$ and condition (b) are incompatible.

Proof. Suppose first that the set W of elements in G generates G modulo $C(G)$, but does not generate G. Denote by V the subgroup generated by W, so that $V<G$. Since G is generated by a finite number of elements, it is possible to generate G by adjoining a finite number of elements to V. Let U be a smallest (finite) set of elements in G so that G is generated by V and U. Since $V<G$, it follows that U does not contain the identity and that it contains at least one element u. Denote by T the subgroup of G which is generated by adjoining all the elements not equal to u in U to V. It follows from the minimum-property of U that $T<G$, and that T does not contain u. There exists a greatest subgroup S of G which contains T but not u. If B is a subgroup of G so that $S<B$, then u is an element in B, that is, $B=G$, so that S is a greatest subgroup of G. Since G is generated by adjoining W to $C(G)$, every subgroup of G that contains both W and $C(G)$ is equal to G. Thus a proper subgroup of G cannot contain both W and $C(G)$. S is a proper subgroup of G which contains $W\leqq V\leqq T\leqq S$. Hence $C(G)$ is not part of S. Hence S is not a normal subgroup of G, since otherwise G/S would be a cyclic group and $C(G)\leqq S$. Thus S is a greatest subgroup of G which is not a normal subgroup of G; and (b) is a consequence of (a).

Suppose now that (b) is satisfied by G and that S is a greatest subgroup of G. Then S cannot generate G modulo $C(G)$, since $S<G$. The subgroup T, generated by S and $C(G)$, is therefore a proper subgroup of G, that is,

* This is a generalization of Burnside's minimal basis theorem for groups of order a power of a prime number; cf. for example, Philip Hall, Proceedings of the London Mathematical Society, (2), vol. 36 (1933), pp. 29–95; in particular, pp. 35–36. Also H. Wielandt, Mathematische Zeitschrift, vol. 41 (1936), pp. 281–282.

$S \leqq T \leqq G$; and since S is a greatest subgroup of G, this implies $S=T$ or $C(G)<S$; and S is consequently a normal subgroup of G, so that (a) is a consequence of (b).

COROLLARY 4.8. *The following property of groups G is a necessary and sufficient condition for the validity of condition* (G):

(G**) *If the subgroup S of G is generated by a finite number of elements, and if the subset U of S generates S modulo C(S), then S is generated by U.*

This is a consequence of Theorems 4.1 and 4.7.

4.D. The following property deserves at least a short discussion in this context, since it may be used as a definition for finite nilpotent groups.

(M) *If S is a subgroup of the group G, and if T is a proper subgroup of S, then there exists an element s in S so that sT = Ts though s is not an element in T.*

In other words, proper subgroups are proper subgroups of their normalizers.

LEMMA 4.9. *If* (M) *is satisfied by the group G, then G is metacyclic and satisfies* (G).

REMARK. Example 3.4 shows that metacyclic groups satisfying (G) need not satisfy (M).

Proof. If S is a proper subgroup of the group G, then there exists in G an element g so that $gS=Sg$ and so that S is a proper subgroup of the group T, generated by adjoining g to S. Clearly S is a normal subgroup of T and T/S is a cyclic group; and this implies the metacyclicity of G. If furthermore K is a greatest subgroup of the subgroup H of G, then the normalizer of K in H is different from K and therefore equal to H, so that K is a normal subgroup of H, that is, (G) is satisfied by G.

COROLLARY 4.10. *A group satisfies* (M) *and does not contain elements of infinite order if, and only if, it is a direct product of p-groups, satisfying* (M).

This is a consequence of Lemma 4.9 and Corollary 4.2, since a metacyclic group without elements of infinite order is by Lemma 3.1 finite, when generated by a finite number of elements.

LEMMA 4.11. *If $G=Z_\zeta(G)$ for some ordinal ζ, then* (M) *is satisfied by the group G.*

Proof. Suppose that S is some subgroup of G, and that T is a proper subgroup of S. There exists by Lemma 2.1 an ordinal σ so that $S=Z_\sigma(S)$; and there exists therefore an ordinal τ so that $Z_\tau(S) \leqq T$ but it is not true that $Z_{\tau+1}(S) \leqq T$. There exists therefore an element v in $Z_{\tau+1}(S)$ which is not con-

tained in T. If g is any element in S, then $vgv^{-1}g^{-1}$ is an element in $Z_\tau(S) \leqq T$; and this implies $vT = Tv$, so that (M) is satisfied by the group G.

COROLLARY 4.12. *Properties* (M) *and* (G) *are equivalent properties of soluble as well as of finite groups.*

This is a fairly immediate consequence of Lemmas 4.9 and 4.11, Theorem 2.2 and Corollary 2.3.

Generalizing a concept introduced by H. Wielandt, op. cit. (second paragraph of the second footnote on page 405), we say that the subgroup S of the group G is a *subnormal subgroup of* G whenever there exists an ascending chain of subgroups B_ν with the following properties:

(i) $B_0 = S$;

(ii) B_ν *is a normal subgroup of* $B_{\nu+1}$;

(iii) B_ν *is for limit-ordinals* ν *the set of all the elements contained in groups* B_μ *for* $\mu < \nu$;

(iv) $B_\gamma = G$.

THEOREM 4.13. *Each of the following three properties of a group* G *implies the others*:

(a) *Condition* (M) *is satisfied by the group* G.

(b) *If* T *is a proper subgroup of* G, *then* T *is a proper subgroup of its normalizer in* G.

(c) *Every subgroup of* G *is a subnormal subgroup of* G.

Proof. Condition (b) is a trivial special case of condition (M), so that (a) implies (b). That (b) implies (c) is a consequence of the fact that every subgroup of G is a normal subgroup of its normalizer in G. Suppose finally that (c) is satisfied by the group G, that S and T are any two subgroups of G and that S is a proper subgroup of T. Since S is a subnormal subgroup of G, there exists an ascending chain of subgroups B_ν, starting with S and satisfying the above conditions (i) to (iv). Denote by T_ν the crosscut of T and B_ν. Then we have $S = T_0$, and there exists some ordinal ρ so that $S = T_\rho < T_{\rho+1}$, since S is a proper subgroup of G, since the B_ν sweep out the whole group G, and since $S = T_\lambda$ whenever λ is a limit-ordinal and $S = T_\mu$ for every $\mu < \lambda$, as follows from (iii). If u is any element in $T_{\rho+1}$, then $uT_\rho u^{-1} = T_\rho$, since u is both an element in T and in $B_{\rho+1}$, so that $uT_\rho u^{-1}$ is both a subgroup of T and, by (ii), of $B_\rho = uB_\rho u^{-1}$. Thus $S = T_\rho$ is both a normal and a proper subgroup of $T_{\rho+1} \leqq T$, so that S is a proper subgroup of its normalizer in T, that is, (a) is a consequence of (c).

We are going to discuss *properties f of group elements* which are subject to the following condition.

(S) *If all the elements of the subgroup U of the group G satisfy property f, if g is an element in G which satisfies property f, and if $gU = Ug$, then every element in the subgroup generated by U and g satisfies property f.*

The two examples of properties satisfying (S) which interest us are:

(a) that of being an element of order a power of the prime number p; and

(b) that of being an element of finite order.

If f is a property of group elements so that (S) is satisfied by f, and if the group unit satisfies f, then there exists always a subgroup all of whose elements satisfy f though it is not a proper subgroup of a subgroup with this property. Such a greatest subgroup of the group G is termed an *f-component of G*.

If U is an f-component of G, and if V is a subgroup of G such that U is a normal subgroup of V, then it follows from (S) that U is exactly the set of all the elements in V which satisfy property f.

The properties f we are going to discuss are subject to another condition.

(J) *If the element g in G satisfies property f, and if α is an automorphism of G, then g^α satisfies property f too.*

If property f satisfies (J), then a characteristic subgroup of G is generated by the elements satisfying f.

LEMMA 4.14. *If the property f of elements in the group G is satisfied by the group unit, if the property f is subject to the conditions* (S) *and* (J), *and if condition* (M) *is satisfied by the group G, then there exists one and only one f-component F of G; and F contains all those elements of G which satisfy property f, so that F is a characteristic subgroup of G.*

Proof. As has been remarked before, there exists always at least one f-component U of G. Denote by U' the normalizer of U in G, and denote by U'' the normalizer of U' in G. Clearly U is a normal subgroup of U', and U' is a normal subgroup of U''. Since U is a normal subgroup of U', it follows from condition (S) that U contains all the elements in U' which satisfy property f; and it follows from (J) that U is a characteristic subgroup of U'. Since U' is a normal subgroup of U'', this implies that U is a normal subgroup of U'', so that U'' is part of the normalizer of U. Hence $U' = U''$. Since however (M) is satisfied by the group G, no proper subgroup of G equals its normalizer in G, so that $U' = G$. Hence U is a normal subgroup of G; and U contains consequently all the elements of property f in G, since U is at the same time an f-component and a normal subgroup of G.

COROLLARY 4.15. *If property* (M) *is satisfied by the group G, then the set $F(G)$ of the elements of finite order in G is a subgroup of G; and $F(G)$ is the direct product of its primary components.*

REMARK. This is not a consequence of Corollary 3.10, since in Corollary 3.10 the group G is supposed to be weakly soluble; and Lemma 4.9 assures only the metacyclicity of the group G. That $F(G)$ is the direct product of its primary components could be derived from Corollary 4.10, since $F(G)$ satisfies (M). But Corollary 4.10 has been proved by reference to an analogous theorem on finite groups whereas the following proof contains a direct proof for the "finite case."

It may be pointed out furthermore that these considerations furnish a simple proof of a theorem proved by H. Wielandt, op. cit. (second paragraph of the second footnote on page 405), since groups satisfying condition (M) are metacyclic, and since metacyclic groups are finite if, and only if, they do not contain elements of infinite order and their length is finite.

Proof. If f is the property of being an element of finite order, then it follows from Lemma 4.14 that there exists one and only one f-component of G; and that this f-component contains all the elements of finite order. Thus $F(G)$ is the uniquely determined f-component of G, and as such $F(G)$ is a subgroup of G. Similarly it follows from Lemma 4.14 that there exists for every prime number p one and only one p-component of G (and of $F(G)$) which contains all the elements of order a power of p in G; and this implies that $F(G)$ is the direct product of its primary components.

4.E. At the end of §2 we mentioned the ascending norm chain of a group G. A more systematic treatment of the condition that $G = N_\nu(G)$ for some suitable ordinal ν may be given here.

One verifies readily that if a group is swept out by its ascending norm chain, then so are its subgroups and quotient groups. It is furthermore not difficult to verify that a group is swept out by its ascending norm chain whenever the group is swept out by its ascending central chain. But it is still an open question whether or not $N(G) \neq 1$ implies $Z(G) \neq 1$. In this respect only the following facts are known.

(a) $Z(G) \leqq N(G)$.

(b) If $Z(G)$ contains elements of infinite order, then $Z(G) = N(G)$.*

(c) If $N(G)$ is hamiltonian, then $Z(G) \neq 1$.†

Since hamiltonian groups are groups with abelian central quotient group, and since the iterated norms are characteristic subgroups, the following is true:

(4.16) *G is weakly soluble, if $G = N_\rho(G)$ for some ordinal ρ.*

It is a consequence of Lemma 2.6 that

* Cf. op. cit. (see the first footnote on page 407) Theorem 3, p. 259.

† R. Baer, Compositio Mathematica, vol. 2 (1935), pp. 241–246.

(4.17) *Condition* (G) *is satisfied by the group* G *whenever* $G=N_\rho(G)$ *for some ordinal* ρ.

As a consequence of these last two statements it may be inferred from Lemma 1.9 and Lemma 1.10 that the following assertion holds true.

(4.18) *If* $G=N_\rho(G)$ *for some ordinal* ρ, *and if* $N(G)$ *is an abelian group of finite rank, then* $Z(G)\neq 1$.

The following fact from the theory of groups with cyclic norm quotient groups will be needed.*

(α) If B is an abelian and normal subgroup of the group G, if G/B is a cyclic group, if $B\leqq N(G)$, if D is the crosscut of B and $Z(G)$, and if $B\neq 1$, then $D\neq 1$ and B/D is a cyclic group.

THEOREM 4.19. *A group* G *which is generated by a finite number of elements satisfies* $G=Z_\zeta(G)$ *for some ordinal* ζ *if, and only if,* $G=N_\rho(G)$ *for some ordinal* ρ.

REMARK. Whether or not the condition that G be generated by a finite number of elements is really needed for the truth of this statement is still an open question. The interest of the above theorem lies in its connection with facts like Corollaries 4.5 and 4.6.

Proof. It is a consequence of facts already related in the course of this discussion, and of Lemma 1.11, that it suffices to prove the following fact.

(4.19.1) *If the group* G *is generated by a finite number of elements, and if* $N(G)$ *is an abelian group of infinite rank, then* $Z(G)\neq 1$.

To prove this statement, let $u_1, \cdots, u_k$ be some finite set of generators of G; and denote by U_i the set of elements y in $N(G)$ so that $yu_i=u_iy$. In applying (α) to the subgroup generated by $N(G)$ and u_i, it follows that $N(G)/U_i$ is a cyclic group. Denote now by V_i the crosscut of the groups U_j for $1\leqq j\leqq i$. Then $V_1=U_1$, so that $N(G)/V_1$ is a cyclic group; and V_{i-1}/V_i is a cyclic group, since V_i is the crosscut of V_{i-1} and U_i. Thus it follows finally that $N(G)/V_k$ is of finite rank. Hence $V_k\neq 1$, since $N(G)$ is of infinite rank. But $V_k\leqq Z(G)$, since G is generated by the elements u_i; and this completes the proof.

(4.20) *If* $G=N_\rho(G)$ *for some ordinal* ρ, *then condition* (M) *is satisfied by the group* G.

Proof. If S is a proper subgroup of G, then there exists an ordinal σ so that $N_\sigma(G)\leqq S$, but so that $N_{\sigma+1}(G)\leqq S$ does not hold. If u is any element in

* Cf. op. cit. (see the first footnote on page 407).

$N_{\sigma+1}(G)$ which is not contained in S, and if s is any element in S, then there exists an integer i and an element t in $N_\sigma(G)$ so that $usu^{-1}=u^it$, and this shows that $uS=Su$. Hence S is a proper subgroup of its normalizer in G; and our statement is a consequence of Theorem 4.13.

(4.21) *If $G=N_\rho(G)$ for some ordinal ρ, then the elements of finite order in G generate a subgroup without elements of infinite order which is the direct product of its primary components.*

This is an immediate consequence of (4.16), (4.17) and of Corollary 3.10.

4.E. If a certain property is not satisfied by the group G, then it may still be possible that G possesses a uniquely determined greatest subgroup with this property. Questions of this type shall occupy us now.

THEOREM 4.22. *In every group G there exists one and only one subgroup $P(G)$ with the following properties:*

(i) *$P(G)$ is a normal subgroup of G;*

(ii) *$P(G)$ is the direct product of its primary components;*

(iii) *if the normal subgroup S of G is the direct product of its primary components, then $S\leqq P(G)$.*

Proof. It clearly suffices to prove the existence of some group $P(G)$.

Suppose first that the normal subgroup S of G is the direct product of its primary components S_p. Then S_p is exactly the set of all the elements of order a power of p in S. Since S is a normal subgroup of G, and since S_p is a characteristic subgroup of S, S_p is a normal subgroup of G. If U is a p-group contained in G, V the subgroup generated by U and S_p, then S_p is a normal subgroup of V, and V/S_p is isomorphic to some subgroup of U. Hence V is a p-group too. This implies in particular that S_p is a subgroup of every greatest p-subgroup of G; or if we denote by $P(G;\ p)$ the crosscut of all the greatest p-subgroups of G, then $S_p\leqq P(G;\ p)$.

$P(G;\ p)$ is by its very construction both a normal subgroup of G and a p-group. The subgroup $P(G)$ of G which is generated by the subgroups $P(G;\ p)$ is therefore a normal subgroup of G and—as is readily verified—the direct product of the $P(G;\ p)$. Thus (i) and (ii) are satisfied by $P(G)$. That (iii) is satisfied by $P(G)$ has been shown in the first part of the proof.

It is an immediate consequence of this theorem that a finite group possesses a uniquely determined greatest normal and nilpotent subgroup.

If $F^*(G)$ is the crosscut of all the greatest subgroups without elements of infinite order, then $F^*(G)$ is a characteristic subgroup of G which does not contain elements of infinite order. If S is a normal subgroup of G which does not contain elements of infinite order, then $S\leqq F^*(G)$, since the subgroup

generated by a subgroup without elements of infinite order and by a normal subgroup without elements of infinite order is itself a subgroup without elements of infinite order.

There exist in every group G normal subgroups S which satisfy $S=Z_\sigma(S)$ for some ordinal σ—for example, $S=1$. Thus the subgroup $R(G)$ of G which is generated by all the normal subgroups S of G which satisfy $S=Z_\sigma(S)$ is a well-determined characteristic subgroup of G. Example 3.4 however shows that $R(G)$ need not satisfy $R(G)=Z_\rho[R(G)]$.

(4.23) *$R(G)$ is weakly soluble.*

If T is a proper subgroup of $R(G)$ and a normal subgroup of G, then there exists a normal subgroup S of G so that $S\leqq T$ does not hold and so that $S=Z_\sigma(S)$ for some ordinal σ. Hence there exists a greatest ordinal τ so that $Z_\tau(S)\leqq T$, and clearly $Z_\tau(S)<S$. The subgroup U which is generated by T and $Z_{\tau+1}(S)$ is a normal subgroup of G, contains T as a proper subgroup, and U/T is abelian. Now it is clear how to finish the proof of (4.23).

It is a consequence of (4.23)—at least if G or $R(G)$ satisfies condition (G)—that the elements of finite order in $R(G)$ generate a subgroup which is a direct product of p-groups (Corollary 3.10).

There exist in every group G normal subgroups S which satisfy

(i) $S=Z_\sigma(S)$ for some ordinal σ;

(ii) $Z_{\nu+1}(S)/Z_\nu(S)$ is an abelian group of finite rank.

The subgroup $R^*(G)$ which is generated by all the normal subgroups S of G that satisfy (i) and (ii) is a well-determined characteristic subgroup of G; and clearly $R^*(G)\leqq R(G)$.

(4.24) *$R^*(G)$ is soluble.*

This statement may be proved in essentially the same manner as (4.23).

It is a consequence of (4.24) and Theorem 2.2 that

$$R^*(G) = Z_\alpha[R^*(G)]$$

for some ordinal α, if only G or $R^*(G)$ satisfies condition (G).

5. In this section we are going to prove the equivalence of the following three properties, provided the groups under consideration are subject to suitable restrictions.

(N) *The subgroup S of the subgroup T of the group G is a normal subgroup of T whenever there exists at most one subgroup B so that $S<B<T$.*

(N*) *Every subgroup of G is a normal subgroup of G.*

(F) *If S is some subgroup of G, and if T is a normal subgroup of S, then S/T is not isomorphic to any of the groups $G_{p,e}$.*

Here p is a prime number, e a non-negative integer and $G_{p,e}$ the group generated by two elements u and v, subject to the relations

(a) u and $c=uvu^{-1}v^{-1}$ are both of order p;

(b) $uc=cu$, $vc=cv$, $v^p=c^e$.

Both the central and the commutator subgroup of this group $G_{p,e}$ are generated by the element c of order p. If $p=2$, then all the groups $G_{2,e}$ are isomorphic; if $p\neq 2$, then $G_{p,0}$ and $G_{p,1}$ are not isomorphic and all the other groups $G_{p,e}$ are isomorphic to one of these two groups.

(5.1) *$G_{p,e}$ satisfies* (G), *but not* (N).

Proof. $G_{p,e}$ is a group of order p^3 and every subgroup of order p^2 contains the element c. Thus (G) is satisfied by $G_{p,e}$.

Denote now by S the subgroup generated by the element u. S is of order p. There exists one and only one subgroup of order p^2 which contains u, namely the subgroup generated by u and c. Thus there exists one and only one subgroup, different from U and G, between U and G. On the other hand U is not a normal subgroup of G, so that (N) is not satisfied by $G_{p,e}$.

LEMMA 5.2. *If G is a group with abelian central quotient group, then each of the three properties* (N), (N*) *and* (F) *implies the others.*

Proof. It is obvious that (N) is a consequence of (N*); and it follows from (5.1) that (F) is implied by (N). Assume now that the group G with abelian central quotient group does not satisfy (N*). Then there exists a pair of elements u, v so that uvu^{-1} is not a power of v. Denote by W the subgroup generated by u and v. The central $Z(W)$ of W contains $c=uvu^{-1}v^{-1}$, and c too is not a power of v. There exists furthermore some prime number p so that c is not a power of c^p. (If c is of infinite order, any p will do; if c is of finite order, any prime divisor p of the order of c may be chosen, since $c\neq 1$.) Since c is an element in the central of W, the subgroup D generated by c^p is a normal subgroup of W. Put $W'=W/D$ and $x'=Dx$ for x in W. Then c' is an element of the exact order p which is contained in $Z(W')$, and c' is not a power of v'. It follows now from the well known properties of groups with abelian central quotient group that $W'/Z(W')$ is a direct product of two cyclic groups of order p, that both u'^p and v'^p are in $Z(W')$, and that $Z(W')$ is generated by c', u'^p and v'^p. Since c' is an element of order p which is not a power of v', $Z(W')$ is the direct product of two groups R' and S' so that R' is cyclic and contains c' and so that S' contains v'^p. If S is the subgroup of W which contains D and satisfies $S'=S/D$, then W'/S' and W/S are essentially the same groups. On the other hand W/S is generated by the elements Su and Sv which are just subject to the defining relations of groups $G_{p,e}$. Hence G does not satisfy (F), so that (F) implies (N*), and this completes the proof.

COROLLARY 5.3. *If the group G satisfies $G=Z_h(G)$ for some integer h, then each of the three properties* (N), (N*) *and* (F) *implies the others.*†

Proof. Again it suffices to show that (N*) is a consequence of (F). This is a direct consequence of Lemma 5.2 if $G=Z_2(G)$. Hence assume that $2<h$, that $Z_{h-1}(G)<G$, and that (F) be satisfied by G. We put $G'=G/Z_{h-3}(G)$. Then $1<Z(G')<Z_2(G')<Z_3(G')=G'$ and (F) is satisfied by G' as well as by $G''=G'/Z(G')$. The latter group is a group with abelian central quotient group, and it follows from Lemma 5.2 that (N*) is satisfied by G''. If u' and v' are any two elements in G', then it follows that $c'=u'v'u'^{-1}v'^{-1}$ is modulo $Z(G')$ both a power of u' and of v', so that there exist integers i, j and elements z, w in $Z(G')$ satisfying $u'^iz=c'=v'^jw$. Since u' permutes with z, and v' with w, it follows that c' permutes with both u' and v'. The group W generated by u' and v' is therefore a group with abelian central quotient group, satisfying (F); and it follows from Lemma 5.2 that W satisfies (N*), so that c' is actually a power of both u' and v'. Consequently G' satisfies (N*), and this implies $G'=Z_2(G')$, a contradiction. This completes the proof.

COROLLARY 5.4. *If $C^i(G)\neq 1$ implies $C^{i+1}(G)<C^i(G)$, then each of the three properties* (N), (N*) *and* (F) *implies the others.*

REMARK. Condition (N*) implies $C^2(G)=1$.

Proof. It suffices again to prove that (N*) is a consequence of (F). Thus assume that (F) is satisfied by G. Put $G'=G/C^3(G)$. Then G' satisfies (F) and $G'=Z_3(G')$; and it follows from Corollary 5.3 that G' satisfies (N*). Hence $C^2(G')=1$ and therefore $C^2(G)=C^3(G)$, and this implies from our general hypothesis concerning G that $C^2(G)=C^3(G)=1$, so that $G=G'$ satisfies (N*).

6. In the discussion of the preceding section condition (G) was always a consequence of the other hypothesis involved. This will be different in the present section. Whenever we assume that a group satisfies both the conditions (G) and (F), we shall express this shortly by saying that the group satisfies condition (F-G).

LEMMA 6.1. *If there exists a normal, abelian subgroup H of G so that G/H is abelian, then each of the three properties* (N), (N*) *and* (F-G) *implies the others.*

Proof. We note first that (N*) implies (N), that (N) implies (G) and, by (5.1), (F), so that (F-G) is a consequence of (N). Hence it suffices to prove that (N*) is a consequence of (F-G).

Let g be some element in G, but not in H; and let h be some element in H. Denote by Q the group generated by g and h. Since $C(Q)\leqq C(G)\leqq H$, it follows that $C(Q)$ is abelian, and that $hc=ch$ for every c in $C(Q)$. Since (G) is satisfied

† This is a special case of Theorem 6.5 below.

by G and by Q, it follows therefore from Lemma 1.7 that $C^i(Q)=1$ or $C^{i+1}(Q)<C^i(Q)$. Hence it follows from Corollary 5.4 that (N*) is satisfied by Q. This implies in particular that either $gh=hg$—in case Q is abelian—or that $ghg^{-1}h^{-1}$ is an element of order 2—in case Q is hamiltonian. If in particular h were an element of order 2, then Q would be abelian: $gh=hg$; for if Q were hamiltonian, $Z(Q)$ would contain the elements of order 2 anyway, so that $gh=hg$ and Q would be abelian.

Since $C(G)\leqq H$, and since H is abelian, it follows from the previous argument that $C^2(G)$ contains only elements of order 1 and 2, and that therefore $C^3(G)=1$. Hence it follows from Corollary 5.4 that (N*) is satisfied by the group G.

COROLLARY 6.2. *If $C_i(G)\neq 1$ implies $C_{i+1}(G)<C_i(G)$, then each of the three properties* (N), (N*), (F-G) *implies the others.*

Note that condition (N*) implies $C_2(G)=1$.

Proof. It suffices again to prove that (*N**) is a consequence of (F-G). Then $G/C_3(G)$, $G/C_2(G)$ and $C(G)/C_3(G)$ satisfy condition (F-G) too. Since the commutator subgroups of both the groups $G/C_2(G)$ and $C(G)/C_3(G)$ are abelian, it follows from Lemma 6.1 that these two quotient groups satisfy (N*). If $G/C_2(G)$ is abelian, then $C(G)=C_2(G)$ and therefore $C(G)=C_2(G)=1$, so that $G=G/C_2(G)$ is abelian. If $G/C_2(G)$ is hamiltonian, then its commutator subgroup $C(G)/C_2(G)$ is of order 2. Since $C_2(G)/C_3(G)$ is the commutator subgroup of $C(G)/C_3(G)$, and since the commutator subgroup of a hamiltonian group is not of index 2 in the hamiltonian group, it follows now that $C(G)/C_3(G)$ is abelian, so that $C_2(G)=C_3(G)$ and therefore $C_2(G)=C_3(G)=1$, so that $G=G/C_2(G)$ is hamiltonian, and this completes the proof.

THEOREM 6.3. *If G is a group without elements of infinite order, then each of the three properties* (N), (N*) *and* (F-G) *implies the others.*

Proof. Again it suffices to derive (N*) from (F-G). We prove first the following somewhat more general statement.

(6.3.1) *If the group H is generated by a finite number of elements of finite order, and if H satisfies* (F-G) (*or* (N)), *then H is a finite group all of whose subgroups are normal.*

It is a consequence of our hypothesis that $H/C(H)$ is a finite group, and it follows from (1.3) that $C(H)$ is generated by a finite number of elements. If H is not abelian, then $C(H)\neq 1$ and it follows from (G) and (1.2) that $C_2(H)<C(H)<H$. But $H/C_2(H)$ is not abelian, and it follows from Lemma 6.1 and (F-G) that $H/C_2(H)$ is hamiltonian. This implies that $H/C_2(H)$ is a finite group, since H is generated by a finite number of elements of finite order.

Hence it follows from (1.3) that $C_2(H)$ is generated by a finite number of elements; and if $C_2(H) \neq 1$, then it follows from (G) that $C_3(H) < C_2(H)$. However it has been proved in Corollary 6.2 that (F-G) and $C_3(H) < C_2(H)$ are incompatible, so that $C_2(H) = 1$, and $H = H/C_2(H)$ is a finite hamiltonian group. This completes the proof of (6.3.1).

If u and v are now any two elements of a group G without elements of infinite order which satisfies (F-G), then the elements u and v generate by (6.3.1) a finite subgroup which satisfies (N*). Hence uvu^{-1} is a power of v, so that G itself satisfies (N*).

COROLLARY 6.4. *If the group G satisfies* (F-G) (*or* (N)), *then the set of elements of finite order in G is a subgroup of G which satisfies* (N*).

This is a consequence of (6.3.1) and Theorem 6.3.

THEOREM 6.5. *If there exists an ordinal* ζ *so that* $G = Z_\zeta(G)$, *then each of the properties* (N*), (N) *and* (F) *implies the others.*

Note that property (N*) implies $G = Z_2(G)$. Whether or not there exist groups G which satisfy (N) (or (F-G)), though not all their subgroups are normal subgroups, is still an open question.

Proof. It suffices to prove that (N*) is a consequence of (F) and $G = Z_\zeta(G)$. It is a consequence of Theorem 2.2 that $G = Z_\zeta(G)$ implies property (G). In order to prove (N*) it suffices to prove that every $Z_\nu(G)$ satisfies (N*). This will be done by complete (transfinite) induction.

$Z_0(G) = 1$ satisfies (N*). Hence we may assume that every $Z_\mu(G)$ for $\mu < \nu$ satisfies (N*).

CASE 1. $\nu = \rho + 1$.

Then $Z_\rho(G)$ satisfies (N*) so that the group $Z_\nu(G) = G'$ satisfies $C_3(G') = 1$, since $C_2[Z_\rho(G)] = 1$, and since $C(G') \leqq Z_\rho(G)$. Hence it follows from Corollary 6.2 that G' satisfies (N*).

CASE 2. ν is a limit-ordinal.

If u and v are two elements in $Z_\nu(G)$, then they are already contained in some $Z_\mu(G)$ for $\mu < \nu$. Since (N*) is satisfied by $Z_\mu(G)$, this implies that uvu^{-1} is a power of v; and consequently $Z_\nu(G)$ satisfies (N*). This completes the proof. Actually we have proved

COROLLARY 6.6. *If the group G satisfies* (N) (*or* (F)), *then every subgroup* $Z_\nu(G)$ *satisfies* (N*).

Appendix. In §§2–4 several conditions have been mentioned which were all of them characteristic properties of nilpotent groups, provided the group in question is a finite group. If it is not assumed that the groups are finite,

then those properties are no longer equivalent. It will therefore be convenient to have a list of these properties together with a chart indicating their interrelations.

(A) *If S and T are subgroups of the group G, and if S is a proper subgroup of T, then there exists a normal subgroup N of T so that $S \leqq N < T$.*

(A′) *If S is a proper subgroup of the group G, then there exists a normal subgroup N of G so that $S \leqq N < G$.*

(G) *If S and T are subgroups of the group G, and if S is a greatest subgroup of T, then S is a normal subgroup of T.*

(G*) *If the subgroup S of the group G is generated by a finite number of elements, and if T is a greatest subgroup of S, then T is a normal subgroup of S.*

(G**) *If the subgroup S of the group is generated by a finite number of elements, and if the subset U of S generates S modulo $C(S)$, then S is generated by the set U.*

(G′) *Every greatest subgroup of the group G is a normal subgroup of G.*

(K) *$G = N_\rho(G)$ for some ordinal ρ.*

(K′) *If the subgroup S of the group G is generated by a finite number of elements, then $S = N_\sigma(S)$ for some ordinal σ.*

(L) *$C^\tau(G) = 1$ for some ordinal τ.*

(M) *If S and T are subgroups of the group G, and if S is a proper subgroup of T, then S is a proper subgroup of the normalizer of S in T.*

(M*) *If S is a proper subgroup of the group G, then S is a proper subgroup of the normalizer of S in G.*

(M**) *Every subgroup of G is a subnormal subgroup of the group G.*

(P) *The group G is a direct product of p-groups.*

(P′) *The elements of finite order in the group G generate a subgroup $F(G)$ of G which does not contain elements of infinite order and which is a direct product of p-groups.*

(R) *If u and v are elements of finite order in the group G, and if the orders of u and v are relatively prime, then $uv = vu$.*

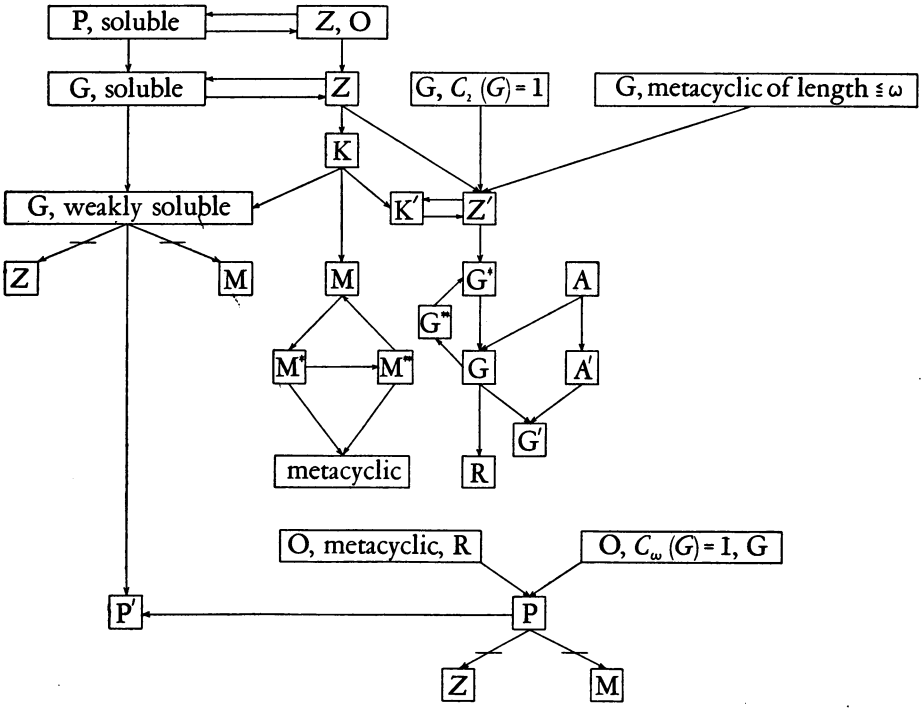
(Z) *$G = Z_\zeta(G)$ for some ordinal ζ.*

(Z′) *If the subgroup S of the group G is generated by a finite number of elements, then $S = Z_\alpha(S)$ for some ordinal α.*

That a finite group, satisfying (L), is a nilpotent group seems to be somewhat accidental if one considers Example 2.5 and the footnote on page 406; and for this reason (L) has not been discussed in this investigation.

In the following chart of inferences we denote by $C_\omega(G)$ the crosscut of the subgroups $C_i(G)$ for integral i; and whenever a group is supposed to consist of elements of finite order only, we indicate this by (O).

Note finally that "arrow crossed out" is used to mean "not implying."



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